3-D Geodetic Inverse

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I. ABSTRACT

Geodetic forward and inverse computations are traditionally computed on the mathematical ellipsoid. Although software is readily available for those computations, the complexity of those geodetic equations can be intimidating. The 3-D global spatial data model (GSDM) provides an alternate method for computing the inverse. Computations in this paper start at a specified point and use numeric integration to establish the latitude and longitude of a second point – both on the ellipsoid. That means the geodesic distance is known and can be used to check the GSDM inverse accuracy. With the latitude and longitude of the endpoints known, the geocentric X/Y/Z coordinates of both points are computed and used to obtain the 3-D spatial separation – i.e., a chord distance. Standard chord/arc equations provide a circular arc distance between the points. For lines of "short" or "moderate" length, this arc distance very nearly approximates the known geodesic distance. A comparison of differences for various geodesic line lengths and configurations is provided. Values for the 3-D azimuth are also compared to geodetic line azimuths computed using Clairaut's Constant. Validity of the 3-D geodetic inverse is established for short and medium length lines. Additional research is needed to establish a reasonable trade-off between possible accuracies for line lengths over about 50 km.

II. INTRODUCTION

A geodetic forward computation starts at a known location (called the standpoint or Pt. 1) on the ellipsoid and traverses to a second location (called the forepoint or Pt. 2) using a geodetic distance and azimuth from the standpoint. A geodetic inverse computation uses the latitude and longitude of both the standpoint and the forepoint to determine the direction and distance between the points. The geodesic distance on the ellipsoid is the same regardless of the direction (Pt. 1 to Pt. 2 or Pt. 2 to Pt. 1) but the geodetic azimuths, forward and back, are different because the meridians at the endpoints are not parallel (unless one point is directly north or south of the other). Various methods are available for computing geodetic forward and inverse positions on the mathematical ellipsoid. One of the most reliable methods is that developed by Vincenty (1975) and used by the National Geodetic Survey (NGS). A numerical integration method was developed by Jank/Kivioja (1980) to exploit computer-based repetitive computations. Accuracy of the iteration method is increased by choosing a shorter element for each iteration. Jank/Kivioja (1980) claim millimeter accuracy for points on opposite sides of the Earth using length elements of 200 meters. A third method of geodetic inverse based upon the geocentric X/Y/Z coordinates of the standpoint and the forepoint is used by Burkholder (in press) to compute combined factors. This paper employs the same method of 3-D inverse but is more generalized by including both azimuth and distance comparisons for inverses for lines up to 50 km in length. The 3-D method is very accurate for short lines (less than 20 km) and the comparisons in this paper can be used to judge the accuracy of results for lines up to 50 km. Additional research is needed to develop valid conclusions for lines over about 50 km.

III. PROCEDURE

Integrity of the comparisons described herein is established by comparing a computed 3-D inverse with known quantities. The known quantities are determined by computing the forward geodetic position of a second point on the ellipsoid using numerical integration and numerous elements over a specified distance and azimuth. The geocentric X/Y/Z coordinates of both the standpoint and the forepoint are computed on the ellipsoid for those computed latitude/longitude positions. The ellipsoid height in each case is zero. The 3-D spatial separation of those points is then computed using the 3-D Pythagorean equation. Since both points are on the ellipsoid the computed 3-D distance is the chord of a circle very closely coincident with the ellipsoid. Over lines of moderate length, the difference between the arc and the chord is quite small. Over longer distances, the difference in length between the geodesic, the chord, and the computed arc becomes apparent.

The forward and the back geodetic azimuths are called the 3-D azimuth (Burkholder 1997) and are computed by rotating the geocentric coordinate differences of the 3-D vector to the local perspective. The standard arctangent function used in plane surveying is used to compute the 3-D azimuth. The 3-D azimuth values are compared to known values – assumed at Pt. 1 and computed at Pt. 2 using Clairaut's Constant. Conclusions are drawn based upon comparisons of known values with computed values for both distances and azimuths.

IV. EQUATIONS

A geodetic forward iteration algorithm patterned after Jank/Kivioja (1980) and described more fully in Burkholder (2008) is used to compute the latitude and longitude of the forepoint from a specified standpoint. Being programmed in a computer, a large number of iterations is inconsequential and the elemental length can be kept quite short in order to insure the accuracy of the forepoint. With the latitude and longitude of both standpoint and forepoint in hand, the geocentric X/Y/Z coordinates are computed for each. Note, east longitude is used in the following equations.

The GRS 1980 ellipsoid is used throughout and given by:

GRS 1980: a = 6,378,137.000 m and e^2 = 0.00669438002290 (1) and (2)

Geocentric X/Y/Z coordinates are computed (ellipsoid height, h, = 0.0) as:

$$N = \frac{a}{\sqrt{1 - e^2 sin^2 ø}}$$ length of normal at ø (3)

$$X = (N + h) \cos ø \cos \lambda$$ geocentric X coordinate (4)

$$Y = (N + h) \cos ø \sin \lambda$$ geocentric Y coordinate (5)

$$Z = [N\,(1 - e^2) + h] \sin ø$$ geocentric Z coordinate (6)

$$D = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}$$ 3-D chord distance (7)

$$\Delta e_{1\rightarrow 2} = -(X_2 - X_1) sin\lambda + (Y_2 - Y_1)\, cos\lambda \qquad (8)$$

$$\Delta n_{1\rightarrow 2} = -(X_2 - X_1)\, sin\o\, cos\lambda - (Y_2 - Y_1)\, sin\o\, sin\lambda + (Z_2 - Z_1)\, sin\o \qquad (9)$$

$$tan\alpha_{1\rightarrow 2} = \Delta e / \Delta n$$ Azimuth Pt. 1 ⟶ Pt. 2 (10)

$$\o_{mean} = \frac{\o_1 + \o_2}{2}$$ Mean latitude (11)

$$R_{mean} = \frac{a\sqrt{1-2}}{(1 - e^2\, sin^2\, \o_{mean})}$$ Radius of curvature at $\o_{mean}$ (12)

$$\theta = 2\, sin^{-1}\left(\frac{D}{2R_{mean}}\right)$$ Subtended angle in radians (13)

$$L = R_{mean}\, \theta$$ Circular arc Pt. 1 to Pt. 2 (14)

$$CC = N_1 cos\o_1 sin\alpha_1$$ Clairaut's Constant (15)

$$\alpha_2 = sin^{-1}\left(\frac{CC}{N_2\, cos\o_2}\right)$$ Geodetic Azimuth at Pt. 2 (16)

V. COMPUTATIONS

The computations for the comparisons listed in Table 1 were performed as follows:

1. The GRS 1980 ellipsoid parameters were used.
2. The east longitude of 240° was arbitrarily selected.
3. The ellipsoid height was taken to be 0.0 in all cases.
4. Computations were performed at 3 different latitudes, 0°, 30°, and 60°.
5. For each latitude, 6 azimuths from Pt. 1 (0°, 15°, 30°, 45°, 60°, and 75°) were computed.
6. For each azimuth case, 4 lengths of geodesic distances were computed, 5K, 10K, 20K, and 50K.
7. Clairaut's Constant is the same numerical value along any geodetic line. As such Clairaut's constant can be used to compute the azimuth of the geodesic at Pt. 2 – see equation (16) above.
8. The back azimuth of a geodetic line at Pt. 2 is the same as the forward azimuth of the geodetic line (from Clairaut's Constant) at Pt. 2 plus 180.°
9. Results of only two cases are tabulated in **Table 1, Comparison of inverse distances and azimuths**.
10. A printout for all 72 cases computed can be viewed at www.globalcogo.com/3D-Inverse.pdf
11. Significant digit considerations include:
    a. Input values are used as "exact."
    b. Each geodesic distance was divided into 10,000 parts for computational precision.
    c. Computed X/Y/Z coordinates in all cases are tabulated to the nearest 0.01 mm. As a practical matter, using X/Y/Z values to the nearest 0.01 mm can probably not be justified. But showing that many decimal places helps control round-off in subsequent computations.
    d. 3-D chord lengths are listed only to the nearest 0.1 mm – 11 significant figures at the most.

e. Local coordinate differences (Δe and Δn) to the nearest 0.1 mm will limit precision of a 3-D azimuth computed from the inverse tangent function.
f. The 3-D azimuths in Table 1 are shown only to the nearest 0.001 seconds of arc.
g. Azimuths at Pt. 1 are "exact" and azimuths at Pt. 2, computed using Clairaut's Constant, are shown to the nearest 0.0001 seconds of arc.
h. When working with very precise quantities over long lines, there is a difference between the azimuth of a geodetic line and the 3-D azimuth at a point. Typically that difference is quite small (< 0.0003 seconds of arc on a 10 km line) as described in Burkholder (1997).

VI. COMPARISONS

Table 1 includes two computational examples intended to show that reliable results for distance and azimuth are easily obtained from geocentric X/Y/Z coordinates of points.

VII. CONCLUSIONS

Conclusions from the comparisons included in this paper include:

1. Reliable geodetic distances and azimuths can be readily obtained from geocentric coordinates for short and medium length lines. Long lines over 50 km deserve additional attention.

2. The 3-D azimuth is "native" to geocentric computations and can be used in place of the azimuth of the geodesic. The exception would be very long lines computed specifically on the ellipsoid.

VIII. REFERENCES

Burkholder, E.F., (in press), "Using the Global Spatial Data Model to compute combined factors," ASCE Journal of Surveying Engineering, Vol. xx, No. yy.

Burkholder, E.F., 2008, The 3-D Global Spatial Data Model: Foundation of the Spatial Data Infrastructure, CRC Press, Boca Raton, FL.

Burkholder, E.F., 1997, "Three-Dimensional Azimuth of GPS Vector," Journal of Surveying Engineering, Vol. 123, No. 4, pp 139-146.

Table 1. Comparison of inverse distances and azimuths

Example 1	Latitude			Longitude			Ellipsoid Ht.	Pt. 1 - X(m)	Pt. 1 - Y(m)	Pt. 1 - Z(m)
Point 1	30	0	0.00000	240	0	0.00000	0.0000 m	-2,764,128.31966	-4,787,610.68829	3,170,373.73529
	Azimuth to Pt. 2 =			15	0	0.00000				

Geodesic to Pt. 2 (m)	Pt. 2 - X(m)	Pt. 2 - Y(m)	Pt. 2 - Z(m)	3-D Chord	Circular Arc	Arc - Chord	Geodesic - Arc
5,000.0000	-2,761,799.34136	-4,786,164.96973	3,174,555.33268	4,999.9999	50,000.0000	0.0001	0.0000
10,000.0000	-2,759,468.66068	-4,784,716.30097	3,178,734.96008	9,999.9990	10,000.0000	0.0010	0.0000
20,000.0000	2,754,802.19799	-4,781,810.11652	3,187,088.29459	19,999.9918	20,000.0000	0.0082	0.0000
50,000.0000	-2,740,762.10400	-4,773,020.85423	3,212,100.75065	49,999.8710	49,999.9995	0.1285	0.0005

	Azimuth Pt. 1 to Pt. 2			3-D Azi. (Δe/Δn)			Clariaut's Constant Azi.		
5,000.0000	15	0	0.0000	15	0	0.000	-	-	-
Back Azi.	-	-	-	195	0	24.169	195	0	24.1684
10,000.0000	15	0	0.0000	15	0	0.000	-	-	-
Back Azi.	-	-	-	195	0	48.390	195	0	48.3899
20,000.0000	15	0	0.0000	15	0	0.000	-	-	-
Back Azi.	-	-	-	195	1	36.992	195	1	36.9921
50,000.0000	15	0	0.0000	15	0	0.003	-	-	-
Back Azi.	-	-	-	195	4	4.083	195	4	4.0803

Example 2	Latitude			Longitude			Ellipsoid Ht.	Pt. 1 - X(m)	Pt. 1 - Y(m)	Pt. 1 - Z(m)
Point 1	60	0	0.00000	240	0	0.00000	0.0000 m	-1,598,552.29348	-2,768,773.79087	5,500,477.13383
	Azimuth to Pt. 2 =			45	0	0.00000				

Geodesic to Pt. 2 (m)	Pt. 2 - X(m)	Pt. 2 - Y(m)	Pt. 2 - Z(m)	3-D Chord	Circular Arc	Arc - Chord	Geodesic - Arc
5,000.0000	-1,593,959.01154	-2,767,889.06027	5,502,243.20619	4,999.9999	50,000.0000	0.0001	0.0000
10,000.0000	-1,589,364.75414	-5,767,002.63582	5,504,005.88866	9,999.9990	10,000.0000	0.0010	0.0000
20,000.0000	-1,580,173.32427	-2,765,224.70754	5,507,521.07966	19,999.9919	20,000.0000	0.0082	0.0000
50,000.0000	-1,552,575.91698	-2,759,850.32779	5,517,985.18496	49,999.8724	50,000.0000	0.1276	0.0000

	Azimuth Pt. 1 to Pt. 2			Azi. from Δe/Δn			Clariaut's Constant Azi.		
5,000.0000	45	0	0.0000	45	0	0.000	-	-	-
Back Azi.	-	-	-	225	3	17.760	225	3	17.7604
10,000.0000	45	0	0.0000	45	0	0.000	-	-	-
Back Azi.	-	-	-	225	6	35.963	225	6	35.9633
20,000.0000	45	0	0.0000	45	0	0.000	-	-	-
Back Azi.	-	-	-	225	13	13.701	225	13	13.7005
50,000.0000	45	0	0.0000	45	0	0.002	-	-	-
Back Azi.	-	-	-	225	33	17.635	225	33	17.6328