Three-Dimensional Azimuth Of GPS Vector

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Abstract: The azimuth of a line is a 2-dimensional (2D) phenomenon whose meaning is determined by its circumstance. With the advent of global positioning system (GPS) surveying and making spatial data computations in a 3D environment, it is convenient to compute azimuth as the arctan ($\Delta e/\Delta n$) where Δe and Δn represent local geodetic horizon components of a 3D vector (GPS baseline) defined by $\Delta X/\Delta Y/\Delta Z$ geocentric coordinate differences. This article describes the geometrical characteristics of such an azimuth, proposes it to be called the 3D azimuth, and shows how the 3D azimuth is related to a geodetic azimuth.

Introduction

In a generic sense, the azimuth of a line is a simple 2D concept. As one attempts to accommodate various physical measurements, coordinate systems, and mathematical models, the simple angle from north must be specifically qualified for an azimuth to be used without ambiguity. With the advent of global positioning system (GPS) surveying and use of the earth-centered earth-fixed (ECEF) geocentric coordinate system, it has become convenient to compute the azimuth from one point to another as α = arctan ($\Delta e/\Delta n$) where Δe and Δn are the local geodetic frame (Soler & Hothem, 1988) components of a 3D vector defined by $\Delta X/\Delta Y/\Delta Z$ geocentric coordinate differences. The purpose of this paper is to describe the geometrical characteristics of such an azimuth, to propose it to be called the 3D azimuth, and to show how the 3D azimuth is related to a geodetic azimuth.

Definitions

Although not as comprehensive, the following are intended to be consistent with current usage and the definitions given for "azimuth", "meridian", and "plane" in standard references such as ACSM/ASCE (1978), NGS (1986), and ASCE/ACSM/ASPRS (1994):

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- An azimuth is the clockwise horizontal angle at a point (standpoint) formed by the meridian through the standpoint and a line from the standpoint to another point (the forepoint). Qualifiers to the word "azimuth" (such as plane, geodetic, astronomic, grid, and magnetic) really apply to characteristics of the reference meridian and/or to definition of the horizontal plane.

- A plane azimuth is a generic designation which implies an assumption of a flat earth (tangent plane perpendicular to the local plumb line) and use of 2D plane Euclidean geometry. Depending upon whether one is using x & y plane coordinates or northings and eastings, a plane azimuth is computed as $\alpha = \arctan(\Delta x/\Delta y)$ or $\arctan(\Delta e/\Delta n)$. A tacit assumption often, but not necessarily, made when using a plane azimuth is that all meridians are parallel to the meridian through some initial point of beginning.

- A geodetic azimuth is an angle in the plane perpendicular to the ellipsoid normal at the standpoint. Acknowledging the earth's surface curves, the geodetic azimuth is defined as the angle between the tangents to the meridian and to the geodetic line (the geodesic) at the standpoint. Furthermore, a geodetic azimuth is always referenced to the geodetic meridian through the standpoint and accommodates the fact that meridians are not parallel. All geodetic meridians converge at the Conventional Terrestrial Pole (CTP) of the Conventional Terrestrial System, Seeber (1993) and Leick (1995).

- An astronomic azimuth is an angle in the plane perpendicular to the plumb line at the standpoint. Unlike geodetic azimuth, the astronomic azimuth is defined as the angle between vertical planes containing the north celestial pole and the observed object. After being corrected for deflection-of-the-vertical and height of the target [called the skew-normal correction by Bomford (1971)], the intersection of the corrected vertical plane through the observed object with the ellipsoid defines a line on the ellipsoid known as the normal section. Separately, due to polar motion, there may be a small difference between the astronomical meridian (referenced to the celestial pole) and the geodetic meridians which converge at the CTP. Depending upon the circumstances and precision needed, these conditions must be considered to obtain a geodetic azimuth from an astronomical azimuth. There is also a very small difference, referred to later as the geodesic correction, between the normal section geodetic azimuth and the geodetic line azimuth from the standpoint to the forepoint.

- A grid azimuth is the name given an angle in the state plane coordinate systems between grid north and the object sighted from the standpoint. Two presumptions of using a grid azimuth are 1) that all grid meridians are parallel to the central meridian of the zone and 2) that the horizontal angle lies in the plane perpendicular to the local ellipsoid normal (if warranted, a Laplace correction is used to accommodate the deflection-of-the-vertical). This grid azimuth is intended to be used with state plane coordinate geometry computations in the same manner as the plane azimuth described earlier.

3D Azimuth

Many surveying computations are performed using either geodetic or grid azimuths. Of these, the geodetic azimuth is more rigorously defined and is used as a computational standard. As described previously, if one uses an observed astronomical azimuth, corrections must be applied to obtain a geodetic azimuth. These corrections are defined in references such as Bomford (1971), Rapp (1979), and Vaníček & Krakiwsky (1986). Similar corrections (Stem 1986) are required to convert a state plane grid azimuth to a geodetic azimuth. The 3D azimuth as described herein can be related to the grid azimuth as both are related to the geodetic azimuth.

The purpose of this paper is to consider the definition and use of azimuths when working with 3D spatial data--specifically with geocentric coordinate differences of the ECEF system. The 3D azimuth is defined as arctan ($\Delta e/\Delta n$) where Δe and Δn represent the local geodetic frame components of a 3D vector defined by $\Delta X/\Delta Y/\Delta Z$ geocentric coordinate differences. The proposed name is simple, unique, descriptive and enjoys concise definition. Soler and Hothem (1988) call it the geodetic azimuth and Seeber (1993) calls it an ellipsoidal azimuth. Although the difference is small, the 3D azimuth is different than either. The geodetic azimuth as defined by the NGS (1986) is the same as the ellipsoidal azimuth as used by Vanícek & Krakiwsky (1986), both of which agree with the definition for geodetic azimuth listed previously. The difference may be inconsequential to most but, unless one is performing classical geodesy computations on the ellipsoid surface, it is probably more appropriate to be using the 3D azimuth than the geodetic azimuth. With increasing use of GPS based measurements for spatial referencing, this author is convinced a less imposing name (which also enjoys precise definition) will gain wider acceptance among spatial data users. The remainder of this paper examines the relationship between the 3D azimuth and a geodetic azimuth between the same two points.

A primary consideration when working with 3D spatial data is the advantage gained by using coordinate differences. A vector in space defined by geocentric coordinate differences is not affected by changing from one rectangular coordinate system to another. The geocentric ECEF rectangular coordinate system is very efficient for database storage and for manipulating spatial data using rules of solid geometry, but most users are more accustomed to working with local coordinate differences. The rotation matrix equations listed in Appendix I provide an efficient method for bidirectional conversion between the two coordinate systems. Once the geocentric $\Delta X/\Delta Y/\Delta Z$ components are rotated to local $\Delta e/\Delta n/\Delta u$ components, the question becomes, "How can the local components of a 3D vector best be described and/or used?" The simple answer is, "the same way plane surveyors use flat earth latitudes, departures and elevation differences."

When the geocentric coordinate differences are rotated into the local geodetic horizon system, two of the local components lie in a tangent plane perpendicular to the ellipsoid normal through the standpoint. (The third "up" component is parallel with the ellipsoid normal.) The horizontal distance from the standpoint to the forepoint in that tangent plane is $\sqrt{(\Delta e^2 + \Delta n^2)}$ and recognized as identical to the horizontal distance plane surveyors have been using for generations. It is also the same as HD(1) as described by Burkholder (1991). The 3D azimuth from the standpoint to the forepoint is arctan ($\Delta e/\Delta n$) and, as described in the following section, differs only slightly from the geodetic azimuth.

When performing 3D spatial data computations using rules of solid geometry the 3D azimuth needs no correction or refinement. But, if one wishes to use a 3D azimuth to perform computations along the geodesic on the ellipsoid, several small corrections are needed to convert a 3D azimuth to a geodetic azimuth. They are a target height correction and the geodesic correction. A logical sequence would be to make the target height correction first to obtain a normal section azimuth from the 3D azimuth, then the geodesic correction is applied to obtain the geodetic line azimuth from the normal section azimuth.

Equations for Target Height Correction (Rapp 1979):

$$\alpha_N = \alpha_{3D} + \Delta\alpha_1 \tag{1}$$

$$\Delta\alpha_1 = \frac{\rho\, h\, e^2 \cos^2\phi_1}{2\, N_1 (1 - e^2)} \left(\sin 2\,\alpha_{3D} - \frac{S}{N_1} \sin\alpha_{3D} \tan\phi_1 \right) \tag{2}$$

Comments about the target height correction are:

1. The sign of the correction $\Delta\alpha_1$ is determined by the sin $2\alpha_{3D}$. The correction is positive if the standpoint is in the NE or SW quadrants and negative if the standpoint is in the SE or NW quadrants.
2. The height of the standpoint is immaterial because the standpoint lies in the vertex of the dihedral angle being considered.
3. For target points on high mountain tops (4,000 m), the maximum correction would be less than 0.5 arc seconds.

Geodesic Correction from Normal Section to the Geodetic Line:

$$\alpha_G = \alpha_N + \Delta\alpha_2 \tag{3}$$

$$\Delta\alpha_2 = -\frac{\rho\, e^2 S^2}{12\, N_1^2} \cos^2\phi_m \sin 2\,\alpha_N \tag{4}$$

Comments about the geodesic correction are:

1. Equation (4) is already an approximation and specifies use of the mean latitude of the line between the standpoint and forepoint. While it is certainly possible to find that latitude, it is reasonable to use the latitude at the standpoint unless a very long line is involved or unless very high accuracy is required. In those cases, the user is advised to consult standard geodetic references such as Rapp (1979) or Vanícek and Krakiwsky (1986).

2. The magnitude of the geodesic correction is quite small and can be ignored in most cases. For example, regardless of standpoint location or azimuth of the line, the geodesic correction will not exceed 0.0003 arc seconds on a 10 km line or 0.03 arc seconds on a 100 km line.

Example I

The first example consists of two points at Klamath Falls, Oregon, home of Oregon's Institute of Technology. Station "Altamont" is a HARN monument located at the Klamath County fair grounds and station "K-785" is a monument on the Oregon Tech campus whose position was observed, computed, and published by the National Geodetic Survey (NGS). A geodetic inverse for the line from Station "Altamont" to “K-785” is azimuth = 327° 50' 23."18, distance = 5,993.7056 m. The geodetic azimuth from "K-785" to "Altamont" is 147° 48' 49."63. The 3D inverse between the two points is:

```
101     STATION K-785
        X = -2490977.0492  LAT (N+S-)   42 15 16.992900
        Y = -4019738.1880  LON (E+W-) -121 47  9.354261
        Z =  4267460.3834  EL HGT          1297.8660 M

         DELTA X/Y/Z      945.7956M   -4536.0463M   -3804.3968M
         DELTA E/N/U     3193.2963M   -5073.5788M     -73.0530M
         LOCAL PLANE INV: DIST =    5994.8598M  AZI. = 147 48 49.69

102     STA ALTAMONT
        X = -2490031.2536  LAT (N+S-)   42 12 32.567851
        Y = -4024274.2343  LON (E+W-) -121 44 50.170528
        Z =  4263655.9866  EL HGT          1227.6330 M

         DELTA X/Y/Z     -945.7956M    4536.0463M    3804.3968M
         DELTA E/N/U    -3191.0300M    5075.0826M      67.4130M
         LOCAL PLANE INV: DIST =    5994.9258M  AZI. = 327 50 23.25

101     STATION K-785
        X = -2490977.0492  LAT (N+S-)   42 15 16.992900
        Y = -4019738.1880  LON (E+W-) -121 47  9.354261
        Z =  4267460.3834  EL HGT          1297.8660 M
```

The following are noted for the 3D inverse shown above:

1. Except for the sign, the $\Delta X/\Delta Y/\Delta Z$ components are identical for each direction.
2. The local horizontal components are similar, but different because the tangent plane at one station differs from the tangent plane at another station.
   ·The horizontal distances (both are correct) agree within 1:90,000.
   ·Each 3D azimuth is with respect to the local geodetic meridian to the CTP.

Comparison of Azimuths - Example I

Station From - To	3D Azimuth from Δe & Δn	Target Hgt Correction	Geodesic Correction	Geodetic Azimuth From 3-D Inverse
Altamont - K-785	327° 50' 23."25	-0."070	0."000	327° 50' 23."18
K-785 - Altamont	147° 48' 49."69	-0."066	0."000	147° 48' 49."62

Example II

The second example represents conditions which were chosen to illustrate features of the 3D azimuth. Point 1 is located in Circleville, Ohio with a hypothetical ellipsoid height of 100 meters. Point 2 is 100 km distant at an azimuth of 45° and at an ellipsoid height of 4,000 m. The geodetic positions and inverse values are:

Point 1: φ = 39° 37' 04."000000; λ = -82° 55' 33."000000; h = 100.000 m.

Point 2: φ = 40° 15' 05."979387; λ = -82° 05' 41."012318; h = 4,000.000 m.

Point 1 to Point 2: Azimuth = 45° 00' 00."000; Distance = 100,000.00 m.

Point 2 to Point 1: Azimuth = 225° 32' 00."649; Distance = 100,000.00 m.

The 3D inverse between the two points is:

```
201    TEST POINT 1
       X =   605912.3508  LAT (N+S-)   39 37  4.000000
       Y = -4882502.1048  LON (E+W-)  -82 55 33.000000
       Z =  4045448.8134  EL HGT           100.0000 M

        DELTA X/Y/Z    64952.7662M   51104.6777M   56487.9348M
        DELTA E/N/U    70752.0653M   70752.2292M    3115.1269M
        LOCAL PLANE INV: DIST =  100058.6462M  AZI. =  44 59 59.76
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```
202    TEST POINT 2
       X =   670865.1170  LAT (N+S-)   40 15  5.979387
       Y = -4831397.4271  LON (E+W-)  -82  5 41.012318
       Z =  4101936.7482  EL HGT          4000.0000 M

        DELTA X/Y/Z   -64952.7662M  -51104.6777M  -56487.9348M
        DELTA E/N/U   -71364.2265M  -70047.4168M   -4684.3645M
        LOCAL PLANE INV: DIST =   99997.4671M  AZI. = 225 32 00.66

201    TEST POINT 1
       X =   605912.3508  LAT (N+S-)   39 37  4.000000
       Y = -4882502.1048  LON (E+W-)  -82 55 33.000000
       Z =  4045448.8134  EL HGT           100.0000 M
```

The following are noted for the 3D inverse shown above:

1. Except for the sign, the $\Delta X/\Delta Y/\Delta Z$ components are identical for each direction.
2. The local horizontal components are very different because of the extreme height difference between the two stations.
   ·The horizontal distances (both are correct) are very different.
   ·Each 3-D azimuth is with respect to the local geodetic meridian to the CTP.

Comparison of Azimuths - Example II

Station From - To	3D Azimuth from Δe & Δn	Target Hgt Correction	Geodesic Correction	Geodetic Azimuth From 3-D Inverse
Point 1 - Point 2	44° 59' 59."76	0."256	-0."016	45° 00' 00."00
Point 2 - Point 1	225° 32' 00."66	0."006	-0."016	225° 32' 00."65

In each of the two examples, the geodetic azimuth computed from the 3D inverse agrees with the geodetic inverse azimuth within 0."01.

Summary and Conclusions:

- Regardless of its name, the 3D azimuth enjoys a simple rigorous definition. As such, its popularity will likely increase as various disciplines implement a 3D spatial data model.

- Given the trend toward use of GPS derived spatial data, the 3D azimuth is likely more appropriate to use than a geodetic azimuth. If a precisely defined geodetic azimuth is needed, it can be obtained readily from a 3D azimuth by applying the target height and geodesic corrections as appropriate.
- The 3D azimuth lies in a plane perpendicular to the ellipsoid normal through the standpoint. Other geodetic meridians in the same plane are not parallel with the geodetic meridian through the standpoint. The difference is convergence.
- The 3D azimuth has a geometrical definition and is not affected by gravity. If angle measurements are made with respect to the local plumb line and if deflection-of-the-vertical is significant, a Laplace correction (as has always been the case) may be appropriate.

Appendix I. Equations of Rotation Matrix (Burkholder 1993):

Local coordinate differences can be computed from geocentric coordinate differences using the matrix formulation (equation 5) or individually using equations 6, 7, and 8.

$$\begin{bmatrix} \Delta e \\ \Delta n \\ \Delta u \end{bmatrix} = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \tag{5}$$

$$\Delta e = -\Delta X\sin\lambda + \Delta Y\cos\lambda \tag{6}$$

$$\Delta n = -\Delta X\sin\phi\cos\lambda - \Delta Y\sin\phi\sin\lambda + \Delta Z\cos\phi \tag{7}$$

$$\Delta u = \Delta X\cos\phi\cos\lambda + \Delta Y\cos\phi\sin\lambda + \Delta Z\sin\phi \tag{8}$$

Geocentric coordinate differences can be computed from local coordinate differences using the matrix formulation (equation 9) or individually using equations 10, 11, and 12.

$$\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = \begin{bmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} \Delta e \\ \Delta n \\ \Delta u \end{bmatrix} \tag{9}$$

$$\Delta X = -\Delta e\sin\lambda - \Delta n\sin\phi\cos\lambda + \Delta u\cos\phi\cos\lambda \tag{10}$$

$$\Delta Y = \Delta e\cos\lambda - \Delta n\sin\phi\sin\lambda + \Delta u\cos\phi\sin\lambda \tag{11}$$

$$\Delta Z = \Delta n\cos\phi + \Delta u\sin\phi \tag{12}$$

Appendix II. Symbols Used in Paper

$\Delta X/\Delta Y/\Delta Z$	=	Coordinate differences in earth-centered earth-fixed geocentric rectangular coordinate system.
$\Delta e/\Delta n/\Delta u$	=	Coordinate differences in local geodetic horizon coordinate system.
α_N	=	Normal section azimuth.
α_G	=	Azimuth of geodetic line (geodesic).
α_{3D}	=	3-D azimuth.
$\Delta\alpha_1$	=	Correction to be applied to 3-D azimuth to obtain normal section azimuth.
$\Delta\alpha_2$	=	Correction applied to normal section azimuth to obtain azimuth of the geodetic line.
ρ	=	206,264.806247096 seconds per radian.
a	=	Semi-major axis of GRS 1980 ellipsoid = 6,378,137.000 m.
e^2	=	Eccentricity squared of GRS 1980 ellipsoid = 0.006694380023.
h	=	Ellipsoid height at forepoint.
φ_1 & φ_2	=	Latitude at standpoint and forepoint respectively.
φ_m	=	Mean latitude between standpoint and forepoint.
λ	=	Geodetic longitude 0° to 360° east. West longitude is negative.
N_1	=	Length of normal at standpoint = $a/\sqrt{(1 - e^2 \sin^2 \varphi_1)}$.
S	=	Distance from standpoint to forepoint = $\sqrt{(\Delta e^2 + \Delta n^2)}$.

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