Algorithms for Low-Distortion Projections

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Definition:

An algorithm is a set of equations used in a computer program to solve a problem. When more than one problem is to be solved, it is proper to use the plural form – algorithms. In this case, an algorithm is the set of equations used for the bidirectional conversion between geodetic coordinates and map projection coordinates. When more than one mapping zone is included, a program may contain several algorithms.

Scope:

This article discusses the algorithms that can be used to perform conversions between geodetic latitude and longitude and state plane coordinates on the new terrestrial reference frame, the North American Terrestrial Reference Frame of 2022 (NATRF2022).[1] Presumably, the National Geodetic Survey (NGS) will publish "official" algorithms in due time. In the meantime, the algorithms listed herein are shown to meet the conversion criterion provided by NGS and are a restatement of the algorithms used on the NAD 83 - modified to accommodate the proposed low-distortion projections (LDPs).

Background:

1. Conformal projections have been used for state plane coordinate systems (SPCS) since the 1930s.
    a. Projection types include . . .
        i.) Lambert conformal conic.
        ii.) Transverse Mercator.
        iii.) Oblique Mercator.
    b. Projections are mathematical but graphic illustrations are used to visualize the model.
    c. Initially, mechanical desktop calculators were used to compute Projection Tables.
    d. Conversion computations included extensive use of logarithms.
    e. Design parameters for SPCS zones did not include elevation – only scale factor.
    f. Combined factor was defined as the product of a sea-level factor times a scale factor.
    g. SPCS length units were feet although NGS has always used meters for geodetic surveying.
    h. In 1959 the 'U.S. Survey Foot' was named for SPCS coordinates until network readjusted.[2]

2. Algorithms for SPCS on NAD 27
    a. Based on Clarke Spheroid of 1866; a = 6,378,206.4 m and b = 6,356,583.8 m.
    b. Publication 62-4 by Charles Claire, State Plane Coordinates by Automatic Data Processing.[3]
    c. Conversion algorithms were designed for electronic computers – more significant digits.
    d. Tailored to duplicate logarithmic computations; absolute within 0.01 foot, relative is better.
    e. In 1964 Prof. Berry designed reference surface for Michigan SPCS at approx. 800 feet.
    f. Berry's algorithm was designed for electronic computation and gave more precise results.
    g. Berry's system was successful although discrepancies in official publications for elevation and grid scale factor computations were confusing to some. See Burkholder's thesis.[4]

3. Algorithms for SPCS on NAD 83
    a. Based on GRS 80 ellipsoid. Parameters: a = 6,378,137 m exact and 1/f = 298.2572221009.
    b. Michigan SPCS for NAD 83 returned to ellipsoid instead of staying at 800 feet elevation.
    c. Burkholder's 1980 Master's Thesis at Purdue University focused on map projections.
    d. Appendix C in Thesis, "The Michigan Scale Factor" was presented at ACSM March 1980.

e. NGS published NAD 83 and some lobbied for continued use of U.S. Survey Foot.
f. Burkholder worked summer of 1983 at NGS to study conversion equations/methods.
g. NGS geodesist, T. Vincenty, summarized conversion equations for all projection types used.
h. NGS Manual 5[5] by J. Stem published in 1989. Burkholder's contribution was acknowledged.
i. Burkholder's algorithm for local coordinate system based on Berry's work & NGS Manual 5.[6]
j. Current low-distortion projections (LDPs) scale projection axis instead of semi-major axis.

North American Terrestrial Reference Frame of 2022 (NATRF2022):

1. Digital revolution means that spatial/geospatial data are digital and 3-D (4-D if time stamped).
2. National Spatial Reference System (NSRS) is being redefined for the spatial data user community.
3. Target date for completion was December 31, 2022, but release date has been delayed.
4. NGS stated policy is to publish horizontal and vertical control values referenced to separate origins.
5. Spatial data user community includes GIS discipline in addition to surveying/engineering/mapping.
6. Geometrical integrity is critical for surveying etc. GIS users are more concerned with uniqueness.
7. Surveyors etc. wish to minimize grid/ground distortion and to be justified in ignoring correction.
8. A low-distortion projection at ground level can be used to minimize distortion for surveyors etc.
9. A state-wide projection at average topography is used for GIS. Distance distortion suffers.
10. Underlying latitude/longitude positions support movement of data from one SPCS zone to another.

Anticipating the Future:

1. NGS will publish results of readjustment
   a. Coordinates
   b. Elevations
   c. Geoid heights
   d. Deflection-of-vertical
   e. Gravity

2. NGS defines "new" SPCS zones
   a. Default policy is continued use of existing zone definitions modified to ground level.
   b. In addition, NGS will design one overall ground-level zone for each state.
   c. Additional zones will be included as needed and/or as requested by local stakeholders.
   d. NGS will define "standard" zones for benefit of the user community.
   e. Each zone will be designed to accommodate elevation/terrain of area covered.
   f. NGS will publish parameters for each SPCS zone.
   g. Primary unit of length is meters. International foot units published at state request.

3. Description of Algorithms for SPCS on NATRF2022:
   a. Projection axis scale is modified to place projection surface closer to topography
      i.) Local
      ii.) Statewide
   b. Changes to map projection model
      i.) Longitude will be counted 0° to 360 ° eastward from the Greenwich Meridian.
      ii.) Lambert – two-standard parallel model replaced by one-standard parallel model.
      iii.) Transverse Mercator – modify defining grid scale factor to accommodate elevation.
      iv.) Oblique Mercator – modify defining grid scale factor to accommodate elevation.
   c. Equations for 2022 conversions will resemble those used in NGS Manual 5.
      i.) Summary of algorithms for three zone types is given in Oregon example.
      ii.) NGS will define all SPCS zones and input parameters

- Stakeholder preferences will be negotiated with NGS state by state.
- Lacking unanimous stakeholder preference, default parameters will be used.

Algorithms for bidirectional conversions:

This section describes algorithms that can be used for performing bidirectional geodetic position and state plane coordinate conversions on the NATRF2022. Each algorithm is generic in that it can also be used for coordinate conversions that are not part of the NGS defined SPCS – UTM for example. In that case, the user is responsible for the input parameters and for the integrity of the results.

In addition to state plane coordinate conversions, the terms “forward” and “inverse” are used to describe geodetic computations on the ellipsoid and in plane surveying computations. To reduce ambiguity, the conversion of latitude/longitude to state plane coordinates (Direct) is designated as BK10 and the conversion of state plane coordinates to latitude/longitude (Inverse) is designated BK11. Likewise, LDPs conversions are designated BK14 (Direct) and BK15 (Inverse) – see Table 1.1 and Figure 1.4 in reference.[7]

The following algorithms are adapted from NGS Manual 5. Those equations were developed over the past 200 plus years by geodesists and mathematicians of long-standing professional stature and current spatial data users enjoy the benefits of their collective contributions. Looking ahead to bidirectional mapping conversions on the NATRF2022, modifications to the Manual 5 algorithms to accommodate LDPs are really quite minimal, but they are enormously important. For those interested, NGS Manual 5 can be downloaded from. . .

https://geodesy.noaa.gov/library/pdfs/NOAA_Manual_NOS_NGS_0005.pdf

With provision for counting longitude eastward, the conversion algorithms for transverse Mercator and oblique Mercator can be used as published in Manual 5.[5] The grid scale factor on the central axis of the zone is already a design parameter. The only change needed for a LDP is that an enlarged grid scale factor is assigned to the central axis of the projection rather than choosing a grid scale factor of unity (1.0) for a tangent projection or less than unity for a typical secant projection (0.9999 for a state plane system or 0.9996 for a UTM projection). The grid scale factor on the axis of a LDP is computed using equation 1.

$$k_0 = 1 + \frac{h_0}{R_G} \qquad (1)$$

where

$k_0 =$ input parameter defining the “scale” of a zone.
$h_0 =$ ellipsoid height used in defining the “scale” of a zone.
$R_G =$ geometrical mean radius of curvature at zone center.

Comments about obtaining a value for k_0:

1. This is a design parameter developed by NGS for the NATRF2022 state plane zones. NGS policy is that k_0 will be computed and held “exact” to 6 decimal places for SPCS zones.
2. k_0 can be computed by user for “any” user-defined application.
3. Based on significant digit considerations, only 3 (maybe 4) digits are needed for h_0 & R_G.
4. For more details on computing k_0, see page 11 of the Oregon Handbook reference.[8]

According to NGS, GRS 80 will be used for NATRF2022 applications and all conversions will be in meters.

1. Semi-major axis a = 6,378,137 m (exact)
2. Flattening 1/f = 298.25722210088. . .
3. Eccentricity squared $e^2 = 0.0066943800229. . .$

Transverse Mercator Projection:

As given in NGS Manual 5, the following parameters are required to define a transverse Mercator projection. The only change needed for a LDP is the defining grid scale factor on the central meridian. These parameters will be determined and provided by NGS for all SPCS zones.

1. Geodetic latitude of the origin.
2. False northing (may be zero) assigned to latitude of origin.
3. East longitude of central meridian (CM), longitude difference is positive east of CM .
4. False easting on central meridian.
5. Grid scale factor on central meridian.

Oblique Mercator Projection:

As given in NGS Manual 5, the following parameters are required to define an oblique Mercator projection. One additional parameter is needed for the oblique Mercator projection because the azimuth of the positive skew axis at the local origin is also required. As before, the only "change" needed for a LDP is the defining grid scale factor at the center (local origin) of the projection. These parameters will be provided by NGS for all SPCS zones.

1. Geodetic latitude of the origin.
2. False northing (may be zero) assigned to latitude of origin.
3. East longitude of central meridian (CM), longitude difference is positive east of CM .
4. False easting on central meridian.
5. Grid scale factor at the local origin.
6. Azimuth of positive skew axis at local origin.

Lambert Conic Conformal Projection:

The two-standard parallel Lambert conic conformal projection is used for SPCS on NAD 27 and NAD 83 projections. The two-standard parallel projection is changed to a one-standard parallel projection on the NATRF2022 datum. That change requires a modification in the order in which derived projection parameters are computed. The underlying equations are the same but, due to a modification in the defining parameters, the mapping radius of the origin, R_0, must be computed before the mapping radius of the equator. Otherwise, the computational procedure is very similar.

The defining parameters for a one-standard parallel Lambert projection are. . .

1. Latitude of central parallel – this may also be latitude of origin.
2. Grid scale factor on projection axis.
3. Latitude of origin.
4. Longitude of central meridian - east
5. False northing on central meridian at grid origin.
6. False easting on central meridian.

When using the two-standard parallel Lambert projection, the latitude of the central parallel, $\emptyset_0$, was determined as a consequence of the defined north and south standard parallels. Using that latitude, the mapping radius of the Equator, K, was computed next. That was the mapping radius used to compute the grid scale factor of the central parallel of latitude. (The computational sequence just described is modified when the grid scale factor of the central parallel is input as a defining parameter.)

On NAD 83, the isometric latitude, Q, (a mathematical requirement) and an intermediate value, W, associated with the various latitudes were computed as:

$$Q_n = 0.5\left[\ln\frac{1+\sin\emptyset_n}{1-\sin\emptyset_n} - e\ln\frac{1+e\sin\emptyset_n}{1-e\sin\emptyset_n}\right] \quad and \quad W_n = \sqrt{1 - e^2\,sin^2\,\emptyset_n} \qquad (2)\ \&\ (3)$$

$$Q_s = 0.5\left[\ln\frac{1+\sin\emptyset_s}{1-\sin\emptyset_s} - e\ln\frac{1+e\sin\emptyset_s}{1-e\sin\emptyset_s}\right] \quad and \quad W_s = \sqrt{1 - e^2\,sin^2\,\emptyset_s} \qquad (4)\ \&\ (5)$$

$$\sin\emptyset_0 = \frac{\ln[W_n\cos\emptyset_s/(W_s\cos\emptyset_n)]}{Q_n - Q_s} \quad and\ K = \frac{a\cos\emptyset_s\exp(Q_s\sin\emptyset_0)}{W_s\sin\emptyset_0} = \frac{a\cos\emptyset_n\exp(Q_n\sin\emptyset_0)}{W_n\sin\emptyset_0} \qquad (6)\ \&\ (7)$$

$$Q_0 = 0.5\left[\ln\frac{1+\sin\emptyset_0}{1-\sin\emptyset_0} - e\ln\frac{1+e\sin\emptyset_0}{1-e\sin\emptyset_0}\right] \quad and \quad W_0 = \sqrt{1 - e^2\,sin^2\,\emptyset_0} \qquad (8)\ \&\ (9)$$

Notes. . .

1. The mapping radius, K, can be computed based on either the north standard parallel or the south standard parallel.
2. The expression exp(x) is the same as ε^x
   where ε = 2.718281828 (the base of natural logarithms).
3. Ln = natural logarithm.
4. $e = eccentricity\ of\ ellipsoid = \sqrt{e^2} = \sqrt{2f - f^2}$

$$R_0 = \frac{K}{\exp(Q_0\sin\emptyset_0)} \quad and \quad k_0 = \frac{(W_0\tan\emptyset_0\,R_0)}{a} \qquad (10)\ \&\ (11)$$

NATRF2022:

Instead of defining the Lambert projection with two standard parallels, this is where the one-parallel Lambert algorithm does it differently. The defining input for the one-parallel projection is the chosen latitude of the central parallel, $\emptyset_0$, and the grid scale factor, k_0, assigned to that latitude. (Other defining parameters are the same.) Remember, a unity (1.0) grid scale factor at the central parallel latitude is associated with a tangent (one-parallel) projection. The conventional Lambert conic conformal secant projection with two standard parallels has a grid scale factor on the central parallel (as computed above) less than unity (1.0). The unique feature of a LDP is a consequence of choosing a grid scale factor greater than unity (1.0) on the central parallel which pushes the projection surface to the ellipsoid height selected by the user.[7] NGS will provide the defining central parallel latitude and the associated grid scale factor for each defined/adopted SPCS zone. Otherwise, equation 1 can be used to determine the defining grid scale factor associated with the selected central parallel latitude.

When defining a one-parallel Lambert projection, equations 10 needs to be used before equation 7 for K. With $\emptyset_0\ and\ k_0$ in hand, equation 12 is solved for R_0 and equation 13 is solved for K as:

$$R_0 = \frac{a\, k_0}{W_0 \tan \emptyset_0} \;\; and \;\; K = R_0 \exp(Q_0 \sin \emptyset_0) = \frac{a\, k_0 \exp\,(Q_0 \sin \emptyset_0)}{W_0 \tan \emptyset_0} \qquad (12)\ \&\ (13)$$

With values for R_0 and K in hand, the remaining computations for the Direct and Inverse conversions proceed as outlined in NGS Manual 5.
Examples:

The Geometronics Unit of the Highway Division of the Oregon Department of Transportation has pioneered use of LDPs in conjunction with NGS by implementing a series of LDPs throughout the state. All three projection types, Lambert Conic Conformal, Transverse Mercator, and Oblique Mercator, are included in the set of 39 different zones covering the entire state. The Oregon Handbook includes computational examples in each of those zones to serve as test cases for those users who wish to program their own conversions. Parameters for each zone are included on pages 23 and 24 of the "Oregon Coordinate Reference System Handbook and Map Set" – version 3.01[8] which can be downloaded as a free pdf file from. . .

ftp://ftp.odot.state.or.us/ORGN/Documents/ocrs_handbook_user_guide.pdf

When using the parameters provided in the Handbook and the algorithms described herein as modified from NGS Manual 5, the results should match the plane coordinate values to 5 decimal places (in meters). The stipulation by NGS is that the bidirectional conversions should be correct within 0.01 mm. Realistically, few users need that level of computational precision but, by imposing that criterion, any discrepancy encountered cannot be blamed on a defective algorithm.

The algorithms described herein were embodied in a program written by the author with results meeting the criterion listed above. A pdf file containing printouts for each example is available at:

www.globalcogo.com/SPCStests.pdf

The actual printouts include the zone parameters in each case and additional results from both Direct (BK14) and Inverse (BK15) conversions. Reader's attention is also directed to an added example in each case showing the latitude and longitude computed from the results of the BK14 conversion. Initially, the input latitude and longitude were input to the nearest 1 second to compute plane coordinates. Then the latitude/longitude (at the conclusion of both a BK14 and subsequent BK15 conversion) should agree within 1.00000 seconds of arc. This agreement – between direct and reverse - demonstrates preservation of the bidirectional computational precision.

References:

1 NGS, 2020, web site on 2022 datums. . . https://www.ngs.noaa.gov/datums/newdatums/index.shtml.
2 Federal Register Notice - 1959, 24 FR 4348.
3 Claire, Charles N., 1968, "State Plane Coordinates by Automatic Data Processing," Publication 62-4.
4 Burkholder, E.F., 1980, "A Metric Map Projection for the State of Michigan," MS Thesis, Purdue.
5 Stem, James E., 1989, "State Plane Coordinate System of 1983," NOAA Manual NOS NGS 5.
6 Burkholder, E.F., 1993, "Design of a Local Coordinate System for Surveying, Engineering, and GIS/LIS," Surveying & Land Information Systems, Vol 53, No. 1.
7 Burkholder, E.F., 2018, *The 3-D Global Spatial Data Model: Principles & Applications, 2n Ed,* CRC Press.
8 Armstrong, M.L., J. Thomas, K. Bays, M. Dennis, 2017, "Oregon Coordinate Reference System – Handbook and Map Set," Version 3.01, Oregon Department of Transportation, Salem, OR.

Addendum to Algorithms Article

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After submitting the first 6 pages of this paper to Benchmarks several weeks ago, I was reminded that a similar listing of the algorithms is included in *The 3-D Global Spatial Data Model: Principles and Applications.* A complete algorithm listing is included in Chapter 11, "Map Projections and State Plane Coordinates." I should have thought of that.

Several comments related thereto:

1. The algorithm in Chapter 11 is specifically applicable to traditional state plane coordinate transformations (NGS says the preferred term is conversions). Minimal changes are needed to those equations for use with low-distortion projections. Those changes are described here in more detail.

2. As big-picture background, the use of state plane coordinates is described in Chapter 1 of the 3-D book. For more information, see Figure 1.4 and Table 1.1. In particular, conversion of latitude and longitude to state plane coordinates and conversion of state plane coordinates to latitude and longitude are identified as BK10 and BK11 computations. Conversions between latitude and longitude and low-distortion projection coordinates are labeled BK14 and BK15 in Figure 1.4. The various conversions between "boxes" are described in Table 1.1.

3. As mentioned in the Benchmarks article, no changes in the algorithms are needed for the transverse Mercator and oblique Mercator projections. For those two projections, the change to a low-distortion projection is accomplished by the user choosing a "different" grid scale factor.

4. The difference for a one-standard parallel Lambert conic projection is that the latitude of the projection axis (central parallel) and associated grid scale factor are selected as defining elements. This is in contrast to a "regular" Lambert conic conformal projection in which two standard parallels are the defining elements. Placement of those two parallels determines both the location of the central parallel (axis of the projection) and the resulting grid scale factor along the projection axis.

5. When using the traditional north and south standard parallels to define a projection, equations 11.25 thru 11.35 in the 3-D book are listed for computing. . .

   a. The latitude of the central parallel, φ_0. — Equation 11.30
   b. The mapping radius of the equator, *K.* — Equation 11.31
   c. The mapping radius of the central parallel R_0. — Equation 11.34
   d. The grid scale factor on the central parallel. — Equation 11.35

6. For a one-standard parallel Lambert conic projection, the latitude of the central parallel and the grid scale factor on that parallel are both defining parameters, leaving the mapping radius of the equator, *K*, and the mapping radius of the central parallel, R_0, as elements to be computed. This is accomplished by first solving equation 11.35 for R_0 , then using that value in equation 11.34 to solve for *K*. Those two computations are listed in the original Benchmarks article as equations 12 and 13.

7. Other than reversal of order in computations described herein, the algorithms as listed in the 3-D book are essentially the same as those given in NOAA Manual NOS NGS 5 (the Bible).