Directions, Angles, and Accuracy

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Nearly everyone uses GPS but understanding angle measurement is still important!

For example, how many (direct/reverse) sets are needed to measure an angle having a standard deviation of 3" or less when using a total station with a least count of 3"?

Answer – It depends!

This article does not take issue with the fact that many surveyors use GPS extensively, but it provides background and procedures by which one can become more confident in the quality of horizontal angles as measured by a theodolite and/or total station. It requires more than claiming "My angles are good because I used a 3-second instrument as required by the specifications." With practical experience one acquires a "feel" for how well the angles are measured. Such experience is valuable and should not be discounted but that experience should be combined with one's knowledge of the capability and limitations of the equipment being used. As a point of reference, early in my career (in the 1970s) I was responsible for computing hundreds of miles of Second-Order control traverses run with 1" theodolites and calibrated EDM. Normal position misclosures were in the 1:30,000 to 1:50,000 range. We routinely observed 4 positions direct/reverse (D/R). Rarely did we need to adjust an observed angle on a control traverse by more than 1 second of arc per station. When warranted (control for a nuclear power plant site) we rented a Wild T-3 theodolite to establish a First-Order network. In that case, the criterion for an astronomical azimuth was 0.45 second of arc. Meeting that and other criteria was an interesting challenge.

Many things have changed since the 1970s but certain underlying concepts do not change and a review of basic concepts is warranted. At the risk of being pedantic, I'll begin with rather obvious issues, then move on to make several points. In many cases, the connotation of the word "error" is bad because it implies something is not correct. In the sense of survey measurements, the word "error" deserves more consideration. Three categories of error in survey measurements include:

1. Mistakes/blunders
2. Systematic error
3. Random error

Mistakes are indeed bad and eliminating mistakes involves "doing the right thing right." This requires knowledge of equipment, procedures, and project/job requirements. Eliminating mistakes also involves checking and rechecking one's work to insure that results are consistent with expectations. In this article, responsibility for and taking steps to eliminate mistakes is left to the user.

Systematic errors arise from imperfections in the equipment being used and/or physical circumstances of a measurement. Systematic errors can be identified, studied, and eliminated by appropriate observing procedures or by computing and applying appropriate corrections. Examples include the angular eccentricity error of a theodolite, maladjustment of a level bubble, and, in taping, the temperature of the tape or the tension applied to a tape during a measurement.

Even if the systematic errors have been identified and eliminated, no measurement is perfect – random errors reflect those imperfections and have three characteristics which, collectively, can provide valuable insight into the quality of a measurement. Random errors (see diagram at http://www.globalcogo.com/SD002.pdf):

1. Are small more often than large – the errors are grouped close to the mean.
2. Are positive and negative in equal numbers – the bell curve is symmetrical.
3. Very large random errors do not occur – they are called "blunders" if they occur.

The modern electronic total station is different than optical theodolites of the past. Even so, with regard to understanding the quality of measured angles, systematic and random errors both need to be considered.

The next two paragraphs involve splitting hairs, but understanding how an angle and a direction are different can be helpful. The difference between an angle and a direction is that a direction is observed while an angle is computed as the difference between two directions. On older instruments such as a transit (having an analog scale), one could actually measure horizontal angles. In cases where the reference direction is zero, the angle and the associated direction have the same numerical value. For example, in those older instruments, the vertical circle typically has "zero" on the horizontal as a reference line – perpendicular to the plumb line. Angles were measured either up or down from the reference horizontal and the reading on the vertical circle was really an observed direction. Because the reference direction is "zero," the direction reading on the vertical circle is called an angle. The angle and direction have the same numerical value. Confused? Maybe the next paragraph will help.

From a statistical perspective, a direction is an independent observation while an angle is not. Computed as the difference between two directions, the uncertainty of an angle is dependent upon the accuracy of the observations on each target. Basic error propagation computations presume independent observations. (In technical terms, independence means that the covariance matrix is diagonal. If the measurements are not independent, the correlation information is carried in the off-diagonal elements of the observational covariance matrix.) That observational independence is a subtle characteristic of optical "directional" theodolites embodying a micrometer for reading both the horizontal and vertical circles. Strictly speaking, it is not possible to measure an angle with a directional instrument – only directions. Angles are computed from directions to the targets. Although less obvious, the angle/direction distinction and error propagation concepts remain valid if using an electronic total station.

A simple exercise has been very useful in an academic setting when studying the characteristics of random and systematic error. The exercise involves use of a well-defined target, a theodolite, and an observer. The activity is observing the direction to the target by pointing the cross-hair on the target and recording the direction.

Two options are:

1. Using an analog micrometer instrument such as a Wild T-2 or Kern DKM-2.
2. Using an electronic total station which displays a unique direction to the target.

The two options are similar but different. A similarity between options 1 and 2 is that direct and reverse observations are employed in each case to cancel out systematic eccentricity error. An important difference is that a direction observation with the older analog instrument has two sources of random error, 1) pointing the cross-hair precisely on the target and 2) setting coincidence on the micrometer prior to reading the scale. The electronic total station eliminates the micrometer setting operation which means the only source of random error is pointing the cross-hair on the target.

For those interested in using option 1, the procedure is slightly more involved because both sources of random error need to be identified. A description of the option 1 procedure is posted at http://www.globalcogo.com/Assign4Pointing.pdf. The remainder of this article considers option 2 – use of an electronic total station.

In addition to the three characteristics of random error described above, several other concepts and equations described in standard surveying texts are:

1. The standard deviation of an observation is computed as:

$$\sigma_{observation} = \sqrt{\frac{\Sigma(\,mean - x_i)^2}{(n-1)}} \quad (1)$$

2. The standard deviation of the mean of repeated observations is:

$$\sigma_{mean} = \frac{\sigma_{observation}}{\sqrt{n}} \quad (2)$$

3. The standard deviation of a series of measurements where multiple error sources all contribute to the uncertainty of the result is:

$$\sigma_{series} = \sqrt{\sigma^2_{mean1} + \sigma^2_{mean2} + \ldots \sigma^2_{mean(n)}} \quad (3)$$

But if the standard deviations of the various parts are identical, then:

$$\sigma_{series} = \sqrt{n(\sigma^2_{mean})} = \sqrt{n}\,\sigma_{mean} \quad (4)$$

Where: x_i = a single observation

$mean$ = the mean of a set of observations of same quantity.

Σ = summation symbol meaning to add individual quantities.

n = number of observations

$\sigma_{observation}$ = standard of a single observation.

σ_{mean} = standard deviation of the mean of a set of observations.

σ_{series} = standard deviation of a collection of pieces.

A formal derivation for equations (2) and (4) can be found at www.globalcogo.com/series.pdf .

A "heads-up" here may be helpful. In the analysis that follows, equations (2) and (4) are each used, effectively canceling out. That leads to a "simple" conclusion that may be misleading.

This paragraph describes an exercise that can be used to document one's finesse with an instrument. The instrument is set up (leveled), a well-defined target is carefully sighted, and the direction is recorded. (Note, the direction can be arbitrary and the value of the recorded direction should be consistent with the least-count of the instrument.) Then, using the horizontal slow motion screw, the cross-hair is moved, and the target is re-sighted. The second recorded direction should be independent of the first. After recording say 25 such observations, the questions to be answered are 1) What is the mean direction to the target? 2) What is the standard deviation of a single pointing? And 3) What is the standard deviation of the mean of the observations? A computational example is posted at www.globalcogo.com/SD001.pdf. A suggestion is to compute the standard deviations in terms of seconds of arc.

Next, plunge the telescope of the instrument, repeat the observations, and compute the mean reverse direction to the target. Also, compute the standard deviation of a single reverse pointing and the standard deviation of the mean of the reverse observations. The operational finesse of the observer is related to the similarity of the standard deviation of a single observation in each direct/reverse data set. The standard deviation of the two means should also be very similar.

For the rest of this article, the standard deviations for single direct and reverse sightings are assumed to be the same. By performing the test described, each person can develop reliable personal data when using a given instrument. To continue, I'll use a personal value of 6" for my ability to point a cross-hair on the target. Backsights and foresights are abbreviated BS and FS.

Given a standard deviation of 6" for each sighting, compute the direction to the BS and the standard deviation of the combined direct/reverse (D/R) direction to the BS as:

$$Mean\ direction\ to\ BS = \frac{direction\ to\ BS(direct)\ +\ \ direction\ to\ BS\ (reverse)}{2}$$

$$\sigma_{BS} = \frac{\sigma_{observation}}{\sqrt{2}} = \frac{6\ seconds}{\sqrt{2}} = 4.2\ sec$$ by equation (2)

If multiple targets are involved, an angle is still the difference between any two named directions. The nomenclature is easier if we consider one direction to be the BS and one direction to be the FS. Then, the computed angle, BS to FS is:

$$Angle\ BS\ to\ FS = direction\ to\ FS - direction\ to\ BS$$

And the standard deviation of one set (D/R) from BS to FS is by series equation (4)

$$\sigma_{angle,1\ set} = \sqrt{\sigma_{BS}^2 + \sigma_{FS}^2} = \sqrt{2\ \sigma_{BS}^2} = \sqrt{2}\sigma_{BS} = \sqrt{2} * 4.2\ sec. = 6\ seconds.$$

And using equation (2) above to solve for n, the number of sets required to end up with a mean angle of 3 seconds is:

$$3\ seconds = \sigma_{angle} = \frac{\sigma_{1\ set}}{\sqrt{number\ of\ sets}} = \frac{6\ seconds}{\sqrt{n}}$$

$$\boldsymbol{Solve\ for\ n}: \sqrt{\boldsymbol{n}} = \frac{\mathbf{6"}}{\mathbf{3"}} = \mathbf{2} \quad \boldsymbol{n} = \mathbf{4} \tag{5}$$

In this case, if my ability to point the cross-hair on the target is 6 seconds, then 4 sets of D/R angles are needed to end up with an angle having a standard deviation of 3 seconds.

Other conditions being met, the take-away from this article is that the number of sets needed to achieve a given level of accuracy (standard deviation) for an angle is:

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$$n\ sets = \left(\frac{personal\ observational\ standard\ deviation\ for\ one\ pointing}{desired\ standard\ deviation\ for\ a\ measured\ angle}\right)^2 \tag{6}$$

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Comments:

1. The answer for the number of sets needed hinges on one's demonstrated finesse in pointing the cross-hair on the target. This value is determined for a specific person-instrument combination. The presumption is that the target is stable, distinct, and well-defined.

2. For the example above, 4 D/R sets are required to meet the 3-second criterion.

3. Each of the 4 sets requires 4 sightings – backsight D/R and foresight D/R.

4. The least count of the instrument being used should be comparable to desired standard deviation of angle being measured. That is, don't expect to be able to use a 10" instrument and many sets to achieve 2" standard deviation for a measured angle. Statistically it might be possible, but such is not good professional practice.

5. Equally important, when running a closed traverse, one should expect the angular misclosure per station to be consistent with the measuring techniques employed.

6. Charles D. Ghilani's book, *Adjustment Computations Spatial Data Analysis 4th* Edition, 2006, John Wiley & Sons, contains additional theory on angle measurement and discusses the DIN 18723 standard for angle measurement.