Accuracy of Elevation Reduction Factor

Earl F. Burkholder,[1] Member, ASCE

Abstract: When using state plane coordinates or performing other geodetic computations, the ratio r/(r+h) is the elevation factor commonly used to reduce a horizontal distance to its sea level or ellipsoidal equivalent. In most cases, an approximation is used for the earth's radius while the value of elevation is usually much better known. That is as it should be. But the question examined in this paper is, "What is an acceptable approximation for the earth's radius and how accurately must the elevation be known to assure sufficient precision in the answer?" An equation that can be used to answer those questions is derived using concepts of error propagation.

Introduction:

A measured slope distance is typically reduced to horizontal using the slope distance and either: 1) the difference in elevation of the end points; or 2) the vertical angle (zenith direction) of the slope distance as measured from either or both ends (Burkholder 1991). Regardless of how the horizontal distance was obtained, the next step is to reduce the horizontal distance either to sea level or, being more specific, to the ellipsoid using the ratio:

$$EF = \frac{r}{r+h} \quad \text{where. . . .} \tag{1}$$

EF = elevation factor,

r = approximate radius of the earth and

h = elevation of horizontal distance being reduced.

[1] Associate Professor, Surveying Engineering Department – 3SUR, New Mexico State University, Las Cruces, New Mexico 88003

The question to be addressed is, “What is an acceptable approximation for the radius of the earth and how accurately must the elevation be known to assure adequate precision in the computed sea level or ellipsoidal distance?” Tools to answers those questions are developed from the following error propagation equation:

$$\sigma_U{}^2 = \left(\frac{\partial U}{\partial X}\right)^2 \sigma_X^2 + \left(\frac{\partial U}{\partial Y}\right)^2 \sigma_Y^2 + \left(\frac{\partial U}{\partial Z}\right)^2 \sigma_Z^2 + ... \qquad (2)$$

where U = f (X, Y, Z . . .); X, Y, and Z are independent variables, and σ_X, σ_Y and σ_Z are the standard deviations of the variables.

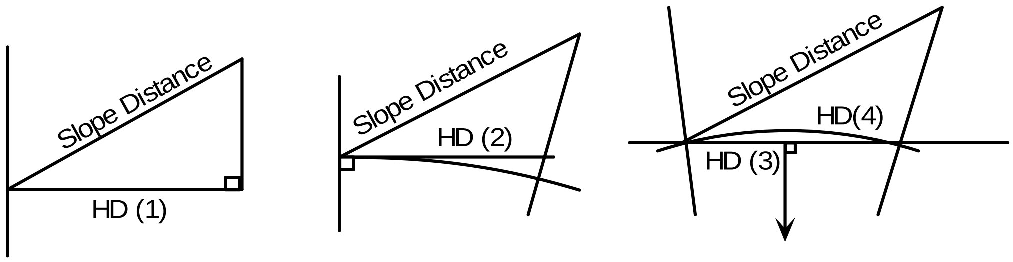
The intent of this article is not to stipulate using one value in preference to another, but to identify a reliable tool that can be used to evaluate the impact of approximations made by the user.

Elements:

The radius of the earth and the elevation of the horizontal distance are the two elements used to reduce a horizontal distance to sea level. It is left for the reader to decide which definition of horizontal distance will be used in the reduction. Of the various definitions of horizontal distance given in Burkholder (1991):

- HD(1), Figure 1a, is the simple right triangle component of slope distance. It is used extensively in practice and is sufficiently precise for many applications.
- HD(2), Figure 1b, is the tangent plane distance between plumb lines.
- HD(3), Figure 1c, is the chord distance between plumb lines. It’s endpoints are at the same elevation and it is perpendicular to the plumb line only at the midpoint.
- HD(4), Figure 1c, is the arc distance between plumb lines at the same elevation as the endpoints of HD(3).

As shown in Burkholder (1991), the differences between HD(2), HD(3), and HD(4) are miniscule. If HD(1) is not precise enough, either HD(2), HD(3), or HD(4) can generally be used instead.



(a) (b) (c)

Figure 1, Slope Distance and Various Definitions of Horizontal Distance

Notes with regard to the elevation factor elements are:

1. When computing the elevation factor, earth's radius is often taken to be 6,372,000 meters or 20,906,000 feet. Either value is acceptable and, as implied by rules of significant digits, is accurate to the nearest 1000 feet or 1000 meters.
2. The earth is not quite spherical, but flattened at the poles. Students of geodesy soon learn that the earth's radius in the north-south direction changes from a smaller value at the equator to a larger value at the poles. The geometrical mean radius of the earth at a given latitude is computed as:

$$r = \frac{a\sqrt{1-e^2}}{1-e^2\sin^2\phi} \qquad \text{where, for GRS 1980 (and NAD83);} \qquad (3)$$

a = semi-major axis (radius of equator) = 6,378,137.000 m

e^2 = eccentricity squared = 0.00669438002290.

ϕ = geodetic latitude.

Table1 lists values of geometrical mean radius for various values of latitude. Increments of 5° are used except in those cases needed to match the approximate values.

Table 1, Geometrical Mean Radius of Curvature at Various Latitudes

Latitude (degrees)	Mean Radius (meters)	Mean Radius (U.S. Survey Feet)
0	6,356,752.314	20,855,444.88
5	6,357,075.580	20,856,505.47
10	6,358,035.749	20,859,655.62
15	6,359,604.205	20,864,801.46
20	6,361,734.148	20,871,789.45
25	6,364,361.913	20,880,410.71
30	6,367,408.778	20,890,406.96
35	6,370,783.223	20,901,477.96
36-43-04	6,372,000.000	20,905,470.00
36-56-36	6,372,161.544	20,906,000.00
40	6,374,383.582	20,913,290.14
45	6,378,101.030	20,925,486.46
50	6,381,822.817	20,937,697.02
55	6,385,435.668	20,949,550.19
60	6,388,829.252	20,960,683.97
65	6,391,899.595	20,970,757.25
70	6,394,552.344	20,979,460.48
75	6,396,705.765	20,986,525.50
80	6,398,293.360	20,991,734.13
85	6,399,266.022	20,994,925.27
90	6,399,593.626	20,996,000.09

3. Figure 2 illustrates the relationship of horizontal distance, sea level distance and ellipsoid distance. It also shows that geoid height is the difference between the ellipsoid and sea level. Historical practice has included reducing horizontal distances to sea level, but when working with state plane coordinates, or other geodetic computations, it is more appropriate to reduce horizontal distance to the ellipsoid. Equation (6), derived in the next section, can be used to assist in deciding

whether it matters if one uses ellipsoid height or orthometric height in computing the elevation factor. Additional material on elevations is found in Burkholder (2002).

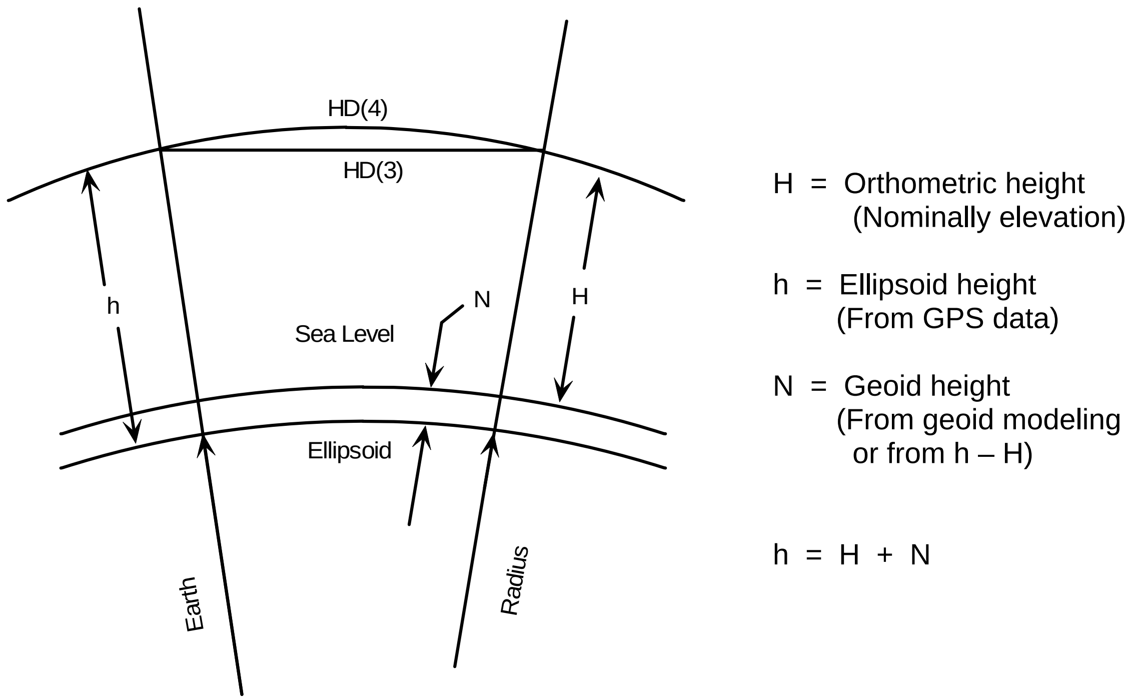


Figure 2 Diagram of Elevation Reduction Factor

Derivation:

The derivation involves taking the partial derivatives of the elevation factor (EF) with respect to each of the two variables, earth radius, r, and elevation, h.

$$EF = \frac{r}{r+h} = r(r+h)^{-1}$$

$$\frac{\partial EF}{\partial r} = (r)(-1)(r+h)^{-2}(1) + \left(\frac{1}{r+h}\right)(1) = \frac{h}{(r+h)^2} \qquad (4)$$

$$\frac{\partial EF}{\partial h} = r(-1)(r+h)^{-2} = \frac{-r}{(r+h)^2} \qquad (5)$$

Now, substituting equations (4) and (5) into equation (2) along with appropriate standard deviations, the equation for the standard deviation of EF becomes:

$$\sigma_{EF}^2 = \left(\frac{\partial EF}{\partial r}\right)^2 \sigma_r^2 + \left(\frac{\partial EF}{\partial h}\right)^2 \sigma_h^2$$

$$\sigma_{EF} = \sqrt{\left[\frac{h}{(r+h)^2}\right]^2 \sigma_r^2 + \left[\frac{-r}{(r+h)^2}\right]^2 \sigma_h^2} \qquad (6)$$

Equation (6) is very powerful in that it can be used to compute the uncertainty (standard deviation) of the elevation factor for any combination of radius and elevation uncertainties selected by the user. Two examples, one in English units and one in metric units, are given next.

English units example:

Assume: r = 20,906,000 feet +/- 1,000 feet

h = 3,280 feet +/- 10 feet

Compute: $EF = \frac{r}{r+h} = \frac{20{,}906{,}000}{20{,}909{,}280} = 0.9998431319$

$$\sigma_{EF}^2 = \left[\frac{3{,}280}{(20{,}909{,}280)^2}\right]^2 (1000)^2 + \left[\frac{-20{,}906{,}000}{(20{,}909{,}280)^2}\right]^2 (10)^2$$

$$\sigma_{EF} = \sqrt{5.62848 * 10^{-17} + 2.28658 * 10^{-13}} = 0.000000478$$

This means that a horizontal distance of 1000.000 feet reduced to the ellipsoid for these conditions computes to be 999.843 feet +/- 0.00048 feet. Certainly the difference between horizontal and ellipsoid distance is significant, but the quality of the computed result does not appear to suffer significantly from using an approximate earth radius (+/- 1,000 feet) and an elevation known only to the nearest 10 feet.

Metric Example:

Assume: r = 6,372,000 meters +/- 1,000 meters

h = 1,000 meters +/- 10 meters

Compute: $EF = \frac{r}{r+h} = \frac{6{,}372{,}000}{6{,}373{,}000} = 0.9998430881$

$$\sigma_{EF}^2 = \left[\frac{1{,}000}{(6{,}373{,}000)^2}\right]^2 (1000)^2 + \left[\frac{-6{,}372{,}000}{(6{,}373{,}000)^2}\right]^2 (10)^2$$

$$\sigma_{EF} = \sqrt{6.06211 * 10^{-16} + 2.46136 * 10^{-12}} = 0.000001569$$

This means that a horizontal distance of 1000.000 meters reduced to sea level for these conditions computes to be 999.843 meters +/- 0.0016 meters. Here too, the difference between horizontal and ellipsoid distance is significant, but the quality of the result does not appear to suffer significantly from using an approximate earth radius (+/- 1,000 meters) and an elevation known only to the nearest 10 meters.

Note in both examples that the contribution of uncertainty due to earth radius (first term under the square root symbol) is much smaller than the contribution due to uncertainty of elevation. Also note, the examples are different in that a 10 meter uncertainty in elevation in the metric example is much larger than the 10 feet uncertainty for elevation in the English unit example.

Tabular Results – Metric:

Table 2 includes representative values of EF at five different elevations for a variety of elevation uncertainties and assuming uncertainty of earth radius from 1,000 m to 20,000 m. The relative uncertainty of the elevation factor for each combination is obtained by computing the reciprocal of the tabular entry. For example, the most precise entry is 0.0000001570 (1:6,369,000) in the upper left corner and the least precise entry is 0.0000157970 (1:63,300) in the lower right corner. Although Table 2 can be quite useful for general circumstances, a specific answer for any combination selected by the user is obtained using equation (6).

Table 2, Standard Deviations of Elevation Factors for Various Combinations

Elev. meters	Elev. Sigma	Radius Sigma 1,000 m	Radius Sigma 5,000 m	Radius Sigma 10,000 m	Radius Sigma 20,000 m
100	1	0.0000001570	0.0000001574	0.0000001589	0.0000001645
100	10	0.0000015693	0.0000015694	0.0000015695	0.0000015701
100	50	0.0000078466	0.0000078466	0.0000078466	0.0000078467
100	100	0.0000156932	0.0000156932	0.0000156932	0.0000156932
500	1	0.0000001574	0.0000001686	0.0000001995	0.0000002920
500	10	0.0000015692	0.0000015703	0.0000015739	0.0000015883
500	50	0.0000078456	0.0000078458	0.0000078466	0.0000078495
500	100	0.0000156912	0.0000156913	0.0000156917	0.0000156931
1000	1	0.0000001588	0.0000001994	0.0000002920	0.0000005168
1000	10	0.0000015691	0.0000015737	0.0000015881	0.0000016443
1000	50	0.0000078444	0.0000078453	0.0000078482	0.0000078598
1000	100	0.0000156888	0.0000156892	0.0000156907	0.0000156965
2000	1	0.0000001644	0.0000002919	0.0000005167	0.0000009970
2000	10	0.0000015692	0.0000015876	0.0000016438	0.0000018518
2000	50	0.0000078421	0.0000078458	0.0000078573	0.0000079035
2000	100	0.0000156839	0.0000156857	0.0000156915	0.0000157147
4000	1	0.0000001851	0.0000005163	0.0000009963	0.0000019741
4000	10	0.0000015705	0.0000016428	0.0000018506	0.0000025158
4000	50	0.0000078376	0.0000078524	0.0000078985	0.0000080803
4000	100	0.0000156743	0.0000156817	0.0000157048	0.0000157970

Conclusions:

Conclusions and final comments are:

1. The average earth radius commonly used is acceptable for all but the most demanding applications. Values of earth radius in Table 1 go from 6,356,752 m at the equator to 6,399,594 m at the pole. Using an average earth radius of 6,372,000 m and an earth radius uncertainty of 20,000 meters, the right-most column of Table 2 includes all latitudes from S 65° to N 65°.
2. The accuracy of the elevation factor is not very sensitive to elevation itself. Table 2 includes elevation factors for elevations of 100, 500, 1000, 2000, and 4000 meters. The differences in each column for a given elevation sigma are quite small.
3. The accuracy of the computed elevation factor is affected by uncertainty in the elevation. But the level of sensitivity appears to be rather low. In fact, unless pursuing extremely accurate results, it appears one can use orthometric heights and ellipsoid heights interchangeably when making the elevation reduction. That means geoid modeling for purposes of elevation reduction is often moot. Table 2 includes and shows the impact of elevation uncertainties up to 100 meters. Few locations in the continental U.S. have geoid heights over 50 meters.
4. Regardless of the values shown in Table 2, the accuracy of the elevation factor for any set of conditions can be readily computed using equation 6. It is probably more important to document the assumptions and values used on a given project or line reduction than it is to use one particular value or another.
5. Using the elevation factor is applying a systematic error correction. If a systematic error is not identified and corrected, the default consequence is treating systematic error as part of the random error budget. This paper treats

the quality of the elevation factor as a random error. Nothing in this paper is meant to suggest that knowledgeable professionals should not evaluate systematic error sources and make the appropriate corrections.

Symbols:

EF = Elevation Factor

r = Radius of earth

h = Elevation of horizontal distance

a = Semi-major axis of earth

e^2 = Eccentricity squared for the earth

ϕ = Geodetic latitude

References:

1. Burkholder, Earl F., 1991; "Computation of Horizontal/Level Distances", Journal of Surveying Engineering, Vol. 117, No. 3, pages 104 - 116.
2. Burkholder, Earl F., 2002, "Elevations and the Global Spatial Data Model (GSDM)," presented at the Institute of Navigation 58th Annual Meeting, June 24-26, 2002, Albuquerque, NM. – www.zianet.com/globalcogo/elevgsdm.pdf