Using GPS to Establish the NAVD88 Elevation on "Reilly" – The A-order HARN Station at NMSU

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Introduction

GPS has become an indispensable tool for establishing horizontal control for many applications. GPS is also used for vertical positioning – especially in the RTK mode – but it is generally conceded that using GPS to establish reliable vertical control is more of a challenge. Station "Reilly" is an A-order HARN monument located in the middle of the NMSU Horseshoe. The NGS publishes very precise geocentric X/Y/Z coordinates for it but the elevation being used for "Reilly" is of lesser quality and of unproven origin. This article describes the method and procedures used to establish an orthometric height on Station "Reilly" using two existing first-order benchmarks in the general area and single frequency GPS. Slightly different (possibly better) results may be obtained using different equipment and/or longer observations, but it was gratifying to obtain results that approach the tolerance for first-order leveling when compared to guidelines for conventional differential leveling.

Equipment and Data Collection

GPS data were collected using three single frequency Trimble 4000SE GPS receivers with identical attachable antennas. One receiver was set up over Station "Reilly" and collected data continuously for two sessions. One receiver each was also set up on Stations "A 245" and "H 245" located 2.5 km and 1.4 km respectively from Station "Reilly." The distance between "A 245" and "H 245" is 1.9 km as shown in Figure 1. The first session ran for 1 hour, then the receivers on "A 245" and "H 245" were swapped in their tribrachs, i.e., the HI's at all three stations remained unchanged for both sessions. The second session also ran for 1 hour, giving 4 independent vectors between 3 stations.

Vectors Components and Covariances

The vectors were processed using the broadcast ephemeris and default processing parameters. The vector components and their covariance matrices are:

<u>Reilly to H 245</u>

			Sxx	Sxy	Sxz
ΔX =	-1,330.994 m	Sxx	1.358685E-07		
ΔY =	112.779 m	Sxy	1.041691E-07	7.253125E-07	
ΔZ =	-450.638 m	Sxz	-7.870650E-08	-3.409353E-07	3.612374E-07

<u>H 245 to A 245</u>

			Sxx	Sxy	Sxz
ΔX =	- 605.974 m	Sxx	5.109368E-07		
ΔY =	1,115.949 m	Sxy	3.647326E-07	2.984444E-06	
ΔZ =	1,420.959 m	Sxz	-3.475972E-07	-1.406505E-06	1.414114E-06

<u>A 245 to H 245</u>

			Sxx	Sxy	Sxz
ΔX =	605.979 m	Sxx	7.547776E-07		
ΔY =	-1,115.954 m	Sxy	8.877959E-07	2.526425E-06	
ΔZ =	-1,420.961 m	Sxz	-2.180677E-07	-8.405698E-07	1.240683E-06

<u>A 245 to Reilly</u>

			Sxx	Sxy	Sxz
ΔX =	1,936.976 m	Sxx	5.991157E-07		
ΔY =	-1,228.722 m	Sxy	1.596318E-06	4.994516E-06	
ΔZ =	-970.326 m	Sxz	-9.123785E-07	-2.859629E-06	1.803680E-06

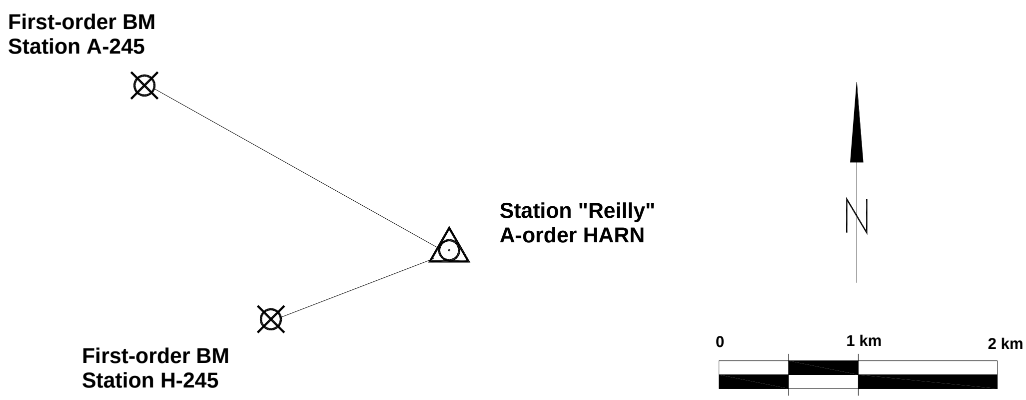

Figure 1 Location of GPS Points

Control Values by the NGS

The NAD83 geocentric X/Y/Z coordinate values for Station "Reilly" and the NAVD88 elevations for stations "A 245" and "H 245" were held fixed during this exercise. The geoid height values at each station as determined by Geoid03 were used to compute the geoid height differences between stations. Those geoid height differences were each assigned an estimated standard deviation of 0.002 meters. With justification, other standard deviations could also be used for the geoid height differences.

Station "Reilly"	X =	-1,556,177.615 m
	Y =	-5,169,235.319 m
	Z =	3,387,551.709 m

Benchmark “A 245”	Elevation = 1,186.626 m
Benchmark “H 245”	Elevation = 1,183.102 m

Geoid Height at (using Geoid03):

Station “Reilly” =	-23.905 m
Station “A 245” =	-23.957 m
Station “H 245” =	-23.954 m

Procedure Used to Determine Orthometric Height

(posted at http://www.zianet.com/globalcogo/gps-elev.htm)

1. Start on 3-D control point with X/Y/Z coordinates and small standard deviations.
2. Collect data (be sure to include HI’s) and build 3-D network with non-trivial vectors.
3. Hold one 3-D point and compute a minimally constrained network.
4. Evaluate and clean up the data. Reject, re-observe, and re-compute as needed.
5. Constrain the network to appropriate 3-D control points. Confirm the fact that no observation is unduly distorted by the adjustment. These X/Y/Z’s are held.
6. Compute latitude/longitude/ellipsoid height (derived quantities) at each point.
7. Identify (valid) elevations at known benchmarks. Compare derived geoid heights with values from geoid model. Investigate discrepancies but don’t change the X/Y/Z’s (unless geoid model or gravity data are more precise than the GPS data).
8. Use Geoid03 (or other geoid model) to determine geoid height **differences** between stations. Combine geoid height differences with GPS derived ellipsoid height differences to get orthometric height (elevation) differences between stations.
9. Compute loop misclosures and misclosures between known benchmarks. These misclosures are then used to assess the quality of elevations obtained using GPS. The elevation of known benchmarks may need to be questioned.

Results of Least Squares Adjustment of GPS Vectors

Several different software packages were used to compute a network adjustment of the observed GPS vectors. Two of them gave identical answers as summarized below.

Station “A 245”

X =	-1,558,114.588 m	+/- 0.0016 m
Y =	-5,168,006.589 m	+/- 0.0042 m
Z =	3,388,522.031 m	+/- 0.0027 m

Station “H 245”

X	=	-1,557,508.610	+/- 0.0012 m
Y	=	-5,169,122.541	+/- 0.0029 m
Z	=	3,387,101.071	+/- 0.0020 m

Geographic Coordinates and Local Standard Deviations

The 3-D coordinate geometry and error propagation software, BURKORD™ (gratis from the author), was used to compute local latitude/longitude/ellipsoid height at each point. Input includes the geocentric X/Y/Z coordinates and the standard deviations at each point in the geocentric reference frame. BURKORD™ output includes local e/n/u standard deviations as well as the latitude/longitude/height at each point. The (derived) results are:

Station “Reilly” (fixed):

Latitude	=	32º 16’ 55.”92906 N	(N) +/- 0.000 m
Longitude	=	106º 45’ 15.”16070 W	(E) +/- 0.000 m
Ellipsoid height	=	1,166.5703 m	(U) +/- 0.000 m

Station “A 245”

Latitude	=	32º 17’ 33.”26476 N	(N) +/- 0.0033 m
Longitude	=	106º 46’ 39.”57110 W	(E) +/- 0.0021 m
Ellipsoid height	=	1,162.6493 m	(U) +/- 0.0038 m

Station “H 245”

Latitude	=	32º 16’ 38.”78107 N	(N) +/- 0.0023 m
Longitude	=	106º 46’ 05.”09688 W	(E) +/- 0.0014 m
Ellipsoid height	=	1,159.1217 m	(U) +/- 0.0026 m

Compare Observed and Modeled Geoid Heights

The definition of geoid height is the ellipsoid height minus known elevation. In this case, each ellipsoid height was obtained from GPS observations and subsequent computations. The known elevation is the elevation published by NGS. The geoid height at each of the two benchmarks is:

			--------- Geoid Height -------		
	Ellipsoid height - elevation	=	Observed	From Geoid03	Diff.
“A 245”:	1,162.6493 m – 1,186.626 m	=	-23.977 m	-23.957 m	-0.020 m
“H 245”:	1,159.1217 m – 1,183.102 m	=	-23.980 m	-23.954 m	-0.026 m

The observed geoid height difference agrees with the Geoid03 ellipsoid height difference within 0.006 meters. That discrepancy is not critical, but worth noting.

Computing Orthometric Height Differences for Each Vector and the Elevation at Station "Reilly"

Although the Geoid03 program provides ellipsoid heights to the nearest millimeter, NGS is careful to state that those values are not accurate within that tolerance. But, geoid height **differences** (based upon the shape of the geoid) are much better. Therefore, the recommended procedure for using GPS to determine elevations is to combine the ellispoid height differences as obtained by GPS with the Geoid03 geoid height differences to obtain orthometric height differences between stations. As shown in Figure 2, the relationship between orthometric height (elevation), ellipsoid height, and geoid height is:

$h = H + N$ where: h = ellipsoid height
H = orthometric height (elevation)
N = geoid height

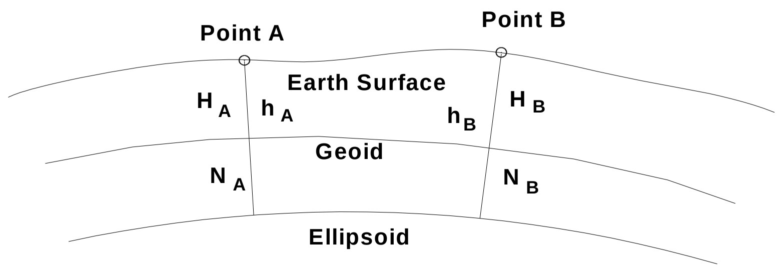

Figure 2 Ellipsoid height, geoid height, and orthometric height

Given that geoid modeling provides better geoid height differences than actual geoid heights, the recommended procedure is to compute the orthometric height (elevation) of Point B from Point A using observed ellipsoid height differences (from GPS) and geoid height differences (from Geoid03) as shown below.

Given: Known elevation at Point A = $H_{A.}$
GPS ellipsoid heights at Points A and B, h_A and h_B.
Geoid03 geoid heights at Points A and B, N_A and N_B.

Find: Elevation (orthometric height) at Point B.

Solution: $\Delta h = h_B - h_A$ (from GPS results)

$\Delta N = N_B - N_A$ (from Geoid03)

$\Delta H = \Delta h - \Delta N$

$H_B = H_A + \Delta H$

Observed Orthometric Height Difference Between Published Benchmarks:

$\Delta h = h_{StaH} - h_{Sta\ A}$ = 1,159.1217 m – 1,162.6493 m = -3.5276 m

$\Delta N = N_{StaH} - N_{StaA}$ = -23.954 m – (-23.957 m) = 0.003 m

$\Delta H = \Delta h - \Delta N$ = -3.5276 m - 0.003 m = -3.531 m

Elevation at Station "Reilly" from Station "A 245"

$\Delta h = h_{Reilly} - h_{Sta\ A}$ = 1,166.5703 m – 1,162.6493 m = 3.9210 m

$\Delta N = N_{Reilly} - N_A$ = -23.905 m – (-23.957 m) = 0.052 m

$\Delta H = \Delta h - \Delta N$ = 3.9210 m - 0.0520 m = 3.869 m

Elevation at Station "Reilly" = 1,186.626 m + 3.869 m = **1,190.495 m**

Elevation at Station "Reilly" from Station "H 245"

$\Delta h = h_{Reilly} - h_{StaA}$ = 1,166.5703 m - 1,159.1217 m = 7.4486 m

$\Delta N = N_{Reilly} - N_A$ = -23.905 m – (-23.954) m = 0.049 m

$\Delta H = \Delta h - \Delta N$ = 7.449 m - 0.049 m = 7.400 m

Elevation at Station "Reilly" = 1,183.102 m + 7.400 m = **1,190.502 m**

The average of the two determinations is: (1,190.495 m + 1,190.502 m)/2 **= 1,190.498 m**

And, the observed loop misclosure based on adjusted GPS observations and Geoid03 modeling is:

$$\text{Misclosure} = \Delta H_{\text{Sta Reilly to Sta H 245}} + \Delta H_{\text{Sta H 245 to Sta A 245}} + \Delta H_{\text{Sta A 245 to Sta Reilly}}$$

$$= -7.400 \text{ m} + 3.531 + 3.869 \text{ m} = 0.000 \text{ m.}$$

The computed misclosure is meaningless because, like inversing an adjusted traverse, the misclosure was obtained from a loop of GPS vectors already adjusted in 3-D space.

What is standard deviation of the computed elevation?

Although more sophisticated methods could be used, the general error propagation equation is used to propogate the error to the final answer. The process is broken into several steps for the sake of simplicity. First, the general error propagation equation is:

$$\sigma_U^{\;2} = \left(\frac{\partial U}{\partial X}\right)^2 \sigma_X^{\;2} + \left(\frac{\partial U}{\partial Y}\right)^2 \sigma_Y^{\;2} + \left(\frac{\partial U}{\partial Z}\right)^2 \sigma_Z^{\;2} + \ldots \qquad \text{where:}$$

σ_U = standard deviation of some computed result.

U = f(X,Y,Z) and

σ_X, σ_Y, and σ_Z are the standard deviations of the variables, X, Y, and Z.

<u>Elevation at “Reilly” from “A 245”:</u>

$El_{Reilly} = El_{StaA} + \Delta h - \Delta N = El_{StaA} + h_{Reilly} - h_{StaA} + \Delta N$

$\sigma_{El\ StaA}$ = 0.000 m (the first-order elevation at Station A 245 is fixed.)
$\sigma_{h\ Reilly}$ = 0.000 (the HARN values are fixed.)
$\sigma_{h\ Sta\ A}$ = 0.0038 m (from least squares adjustment – the “up” component)
$\sigma_{\Delta N}$ = 0.002 m (assumed valid for geoid height difference.)

$\partial El_{Reilly} / \partial El_{StaA} = 1$ $\qquad$ $\partial El_{Reilly} / \partial h_{Reilly} = 1$

$\partial El_{Reilly} / \partial h_{StaA} = -1$ $\qquad$ $\partial El_{Reilly} / \partial \Delta N = 1$

The standard deviation squared of the elevation at Station “Reilly” from Station A 245 is:

$$\sigma^2 = (1)^2 * 0.0^2 + (1)^2 * 0.0^2 + (-1)^2 * 0.0038^2 + (1)^2 * 0.002^2 = 0.00001844\ m^2$$

and the standard deviation of the elevation at Station “Reilly” from “A 245” = **0.0043 m**

<u>Elevation at “Reilly” from “H 245”:</u>

$El_{Reilly} = El_{StaH} + \Delta h - \Delta N = El_{StaH} + h_{Reilly} - h_{StaH} + \Delta N$

$\sigma_{El\ StaH}$ = 0.000 m (the first-order elevation at Station H 245 is fixed.)
$\sigma_{h\ Reilly}$ = 0.000 (the HARN values are fixed.)
$\sigma_{h\ Sta\ H}$ = 0.0026 m (from least squares adjustment – the “up” component)
$\sigma_{\Delta N}$ = 0.002 m (assumed valid for geoid height difference.)

$\partial El_{Reilly} / \partial El_{Sta\ H} = 1$ $\qquad$ $\partial El_{Reilly} / \partial h_{Reilly} = 1$

$\partial El_{Reilly} / \partial h_{Sta\ H} = -1$ $\qquad$ $\partial El_{Reilly} / \partial \Delta N = 1$

The standard deviation squared of the elevation at Station "Reilly" from Station H 245 is:

$$\sigma^2 = (1)^2 * 0.0^2 + (1)^2 * 0.0^2 + (-1)^2 * 0.0026^2 + (1)^2 * 0.002^2 = 0.00001076 \text{ m}^2$$

and the standard deviation of the elevation at Station "Reilly" from "A 245" **= 0.0033 m**

Now, using the general error propagation equation once more to find the standard deviation of the mean elevation at Station "Reilly", the equation for the mean is:

$$Mean\ Elevation = \frac{El\ \ from\ A245 + El\ \ from\ H245}{2} \qquad = \qquad \mathbf{1{,}190.498\ m}$$

and the general error propagation equation elements are:

$$\frac{\partial Mean\ Elevation}{\partial El\ \ from\ A245} = \frac{1}{2} \quad \text{and the} \quad \frac{\partial\ Mean\ Elevation}{\partial El\ \ from\ H245} = \frac{1}{2}$$

Therefore, the standard deviation squared of the mean elevation at "Reilly" and the standard deviation of the mean are:

$$\sigma^2 = \left(\frac{\partial Mean\ Elevation}{\partial El\ \ from\ A245}\right)^2 \sigma^2{}_{El\ from\ A245} + \left(\frac{\partial Mean\ Elevation}{\partial El\ \ from\ H245}\right)^2 \sigma^2{}_{El\ from\ H245}$$

$$\sigma^2 = \left(\frac{1}{2}\right)^2 0.0043^2 + \left(\frac{1}{2}\right)^2 0.0033^2 = 0.000007345 \text{ m}^2 \quad \text{and} \quad \boldsymbol{\sigma} = \mathbf{0.0027\ m.}$$

Comment on Comparison With Leveling Standards

The 1984 Federal Geodetic Control Committee standards for conventional leveling are applicable for either a loop or a section run forward and back. The following comparison may not be valid because GPS is different than conventional leveling. Other assumptions could also be used. For example, the sum of the 3 GPS baseline lengths could be used as the loop distance but as shown earlier, the loop misclosure is zero. However, if we use the standard deviation of the mean elevation as the misclosure (0.0027 m), we should multiply that by 2 so we can make comparisons at the 95% confidence level. And if we use the closest first-order benchmark as the distance (1.4 km), the computed coefficient is:

Allowable misclosure = coefficient (X) √(distance in kilometers) or in this case,

$$5.4\ mm = X\sqrt{1.4} \quad \text{and } X = \frac{5.4}{\sqrt{1.4}} = \quad 4.6\ mm$$

The standards for differential leveling for various orders and classes given in the 1984 FGCC Standards are:

Allowable misclosures:

First-order, class I	3 mm √(distance in kilometers)
First-order, Class II	4 mm √(distance in kilometers)
Second-order, Class I	6 mm √(distance in kilometers)
Second-order, Class II	8 mm √(distance in kilometers)

Using the 1984 FGCC standards, the statement is made that the results given here approach first-order quality. But, according to the more recent standards for vertical control adopted by the Federal Geographic Data Committee and the Federal Geodetic Control Subcommittee, the NAVD88 elevation at Station "Reilly" qualifies as a 5 mm elevation. Of course, other criteria such as equipment used and documented observing procedures must also be met for the NAVD88 elevation to be "approved" by NGS. As noted on page 2-3 of "Geospatial Positioning Accuracy Standards – Part 2: Standards for Geodetic Networks"

http://www.fgdc.gov/publications/documents/standards/endorsed.html

the current accuracy standards at the 95% confidence level for horizontal, ellipsoid height, and orthometric height are as listed below.

Accuracy Classification	95% Confidence, < or = to:
1- millimeter	0.001 meters
2-millimeter	0.002 meters
5-millimeter	0.005 meters
1-centimeter	0.010 meters
2-centimeter	0.020 meters
5-centimeter	0.050 meters

Comparison with previous NGVD29 values converted to NAVD88 using CORPSCON

The NGVD29 benchmark value currently used on Station "Reilly" is 3,904.083 U.S. Survey feet. Using CORPSCON to convert the NGVD29 value to NAVD88 and meter units gives an answer of NAVD88 elevation in meters = 1,190.506 meters. That value is within 0.008 meters of the answer obtained using GPS and two first-order benchmarks in the general area. The consistency of the results is quite gratifying.

Possible Improvements Include

Better results for the elevation of Station "Reilly" could probably be realized if:

1. Dual frequency GPS equipment had been used to collect the data.
2. The observation time would have been longer than two sessions of 60 minutes each.
3. Three or more first-order benchmarks had been used rather than just two.
4. Greater care were taken in making the antenna height measurement. Antenna height measurements in this exercise were good to about 1-2 mm.
5. The precise ephemeris would have been used in processing the vectors. Since the vectors were not very long, using a precise ephemeris would probably make little difference.

Conclusions

1. Reliable elevations can be obtained using GPS. But, the process is tedious.
2. Geoid 03 can easily be used to the surveyors' advantage.
3. The NGVD29 elevation of Station "Reilly" was really pretty good.
4. At least in this case, an acceptable NAVD88 elevation of Station "Reilly" could have been obtained using CORPSCON (really the NGS VERTCON) program. But, the GPS results were needed to validate that conclusion.
5. The antenna height measurement is critical and possibly the weakest part of process.
6. Using identical model antennas obviates the need to know the exact measurement to the antenna phase center.