Standard Deviation and Network/Local Accuracy of Geospatial Data

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Abstract:

The Discussion/Closure items published in the February 2012 issue of the ASCE Journal of Surveying Engineering address a difference of opinion that exists with regard to computing local accuracy of geospatial data. The Discussion focuses on whether one or two rotation matrices should be used when computing local accuracy. After reading the Closure, it appears that a more fundamental question is whether or not a standard deviation can be used as a measure of local accuracy. If it is determined that local accuracy cannot be computed in terms of a standard deviation of the separation between points, then this response is moot. If it can, then a separate question needs to be considered - is local accuracy to be considered as an attribute of a pair of points in 3-D space or is local accuracy an attribute of the distance between two points? That distinction is important because each answer implies a different functional model. It seems that authors in the Discussion/Closure talk past each other on that issue.

This paper provides additional insight by computing the standard deviation of a 3-D slope distance with the hypothesis that a distance standard deviation is an acceptable measure of accuracy. The advantage is that the 3-D computation uses no rotation matrices – it is an independent computation. Standard deviations of short (1 km), medium (20km) and long (100 km) lines are computed using one rotation matrix, two rotation matrices, and no rotation matrix. Comparison of the results between methods shows very little difference in the computed standard deviations. Additionally, all three methods show the same (significant) improvement of results due to using the off-diagonal cross-correlation portions of the overall covariance matrix – local accuracy is better than network accuracy. The take-away is that use of the stochastic model for local accuracy in Burkholder (2008) is validated and that the relative location of one point with respect to another (network accuracy) can be computed using the covariance matrices of the two points (no statistical correlation between points) but that a better answer (local accuracy) is computed using the full covariance matrix between the two points.

Introduction:

The goal in writing the book, *The 3-D Global Spatial Data Model* (Burkholder 2008), was to start with the assumption of a single origin (Earth's center of mass) for 3-D geospatial data and to use the Earth-centered, Earth-fixed (ECEF) rectangular geocentric coordinate system as the basis of an efficient model for handling 3-D geospatial data. In the ECEF environment, long-standing rules of solid geometry (functional model equations) are used to compute X/Y/Z positions in three-dimensional space. Equivalent positions in other coordinate systems can be obtained using traditional transformation equations found in Burkholder (2008) and in various texts. Standard deviations of derived quantities are computed using stochastic model equations – in this case, the matrix formulation of error propagation given by:

$$\boldsymbol{\Sigma}_{YY} = \boldsymbol{J}_{YX}\, \boldsymbol{\Sigma}_{XX}\, \boldsymbol{J}_{XY}^{T} \qquad (1)$$

Where $\boldsymbol{\Sigma}_{YY}$ is the covariance matrix of the computed result, $\boldsymbol{\Sigma}_{XX}$ is the covariance matrix of the variables, and $\boldsymbol{J}_{YX}$ is the Jacobian matrix of partial derivatives of the computed result with respect to the variables.

Following-the-evidence and developing efficient procedures for handling spatial data were the intent when writing the 3-D book. Computational results (e.g., geodetic forward/inverse computations and network adjustments) can often be achieved more easily using the 3-D formulation rather than traditional geodesy equations. In the case of spatial data accuracy, standard deviations were used by Burkholder (1999) as the basis for computing what is called network accuracy and local accuracy. The extension from computing a standard deviation to calling it network or local accuracy seems reasonable and provides a concise mathematical basis for those accuracy definitions. However Soler & Smith (2010) provide an alternative rigorous formulation for computing local accuracy that is more general and uses the positional errors at each point as the basis of the functional model instead of the inverse distance. Their implication is that local accuracy as presented in Chapter 11 of Burkholder (2008) lacks appropriate rigor. That implication is made specific following equation (13) in Soler/Han/Smith (2012) when they state that the approximation used by Burkholder "is not a general local accuracy estimate and may be applied only when the two points are located very close to one another." Later in the same paragraph they continue "…it is not an estimate of local accuracy as currently defined." Ironically, the comparisons tabulated herein show very little difference in the accuracies computed using one or two rotation matrices, even for points separated by 100 km. While there may be a technical difference in the definitions of "standard deviation" and "local accuracy," computation of the standard deviation of a 3-D slope distance (that is nearly horizontal) and comparing those results with results obtained using either one or two rotation matrices validates use of the stochastic model in Burkholder (2008).

This paper provides additional detail by computing the standard deviation of a 3-D mark-to-mark (slope) distance between points. The reason for using this approach is that the standard deviation of the 3-D slope distance can be computed from the equation of the 3-D inverse distance between points without using a rotation matrix. But, this point must also be clear – the standard deviation of the slope distance is not the same as the standard deviation of the horizontal distance between points. Functional model equations for horizontal distance are different from the functional model equations for slope distance. However, in cases where the horizontal distance agrees closely with the 3-D slope distance, the results of the methods being compared should be nearly identical. Specifically, each of the three methods compared herein uses a different functional model equation but the same covariance matrix between points is used in all cases.

The short (0.968 km) line cited herein is the example given in Chapter 11 of Burkholder (2008). The medium (21.87 km) and long (99.78 km) lines are a portion of the GPS CORS (WISCORS) network in Southeastern Wisconsin – see Figure 1. RINEX data were downloaded from 9 CORS stations and 16 non-trivial baselines were computed (ΔX, ΔY, ΔZ along with baseline covariance matrices) using off-the-shelf software. Standard matrix manipulation software was used to compute the least squares adjustment of the network from which the covariance matrices of the computed positions were obtained. Note that although the computed coordinates of the network agreed very closely with the adopted CORS values, duplicating those ECEF coordinate values was not the objective. The (successful) objective was obtaining the covariances of the computed X/Y/Z coordinates of each point in the example network along with the correlation sub-matrices between points.

Functional Model:

The functional model equation used to compute the slope distance between points is obtained from ECEF coordinates as:

$$D = \sqrt{\Delta X^2 + \Delta Y^2 + \Delta Z^2} \qquad (2)$$

Where D = the 3-D slope distance, $\Delta X = X_2 - X_1$, $\Delta Y = Y_2 - Y_1$, and $\Delta Z = Z_2$ - Z1. Note that the distance in equation (2) is the same whether going from Point 1 to Point 2 or from Point 2 to Point 1, insuring the existence of the commutative property for the standard deviation of the slope distance between points.

Solution:

This derivation uses the ECEF coordinates for each point, the covariance matrix at each point, the cross covariances between points, and the error propagation procedure as defined by equation (1). In this case, the answer will be a 1 x 1 matrix representing the variance of the 3-D spatial distance. Standard deviation is the square root of the variance. In order to use equation (1), the partial derivatives of the inverse distance with respect to each coordinate variable need to be computed and the matrix multiplications need to be performed. The Jacobian matrix (in this case a vector) of partial derivatives of the 3-D spatial distance with respect to each ECEF coordinate value is given by equation (3) as:

$$\boldsymbol{J} = \left[\frac{\partial D}{\partial X_1} \; \frac{\partial D}{\partial Y_1} \; \frac{\partial D}{\partial Z_1} \; \frac{\partial D}{\partial X_2} \; \frac{\partial D}{\partial Y_2} \; \frac{\partial D}{\partial Z_2} \right] = \left[\frac{-\Delta X}{D} \; \frac{-\Delta Y}{D} \; \frac{-\Delta Z}{D} \; \frac{\Delta X}{D} \; \frac{\Delta Y}{D} \; \frac{\Delta Z}{D} \right]. \qquad (3)$$

The covariance matrix of the variables, $\boldsymbol{\Sigma_{XX}}$, equation (4), adopts the labeling convention used by Soler/Smith (2010) and includes both the covariance matrix for each point and the cross covariance matrices between the points.

$$\Sigma_{XX} = \begin{bmatrix} \Sigma_{11} = \begin{bmatrix} \sigma_{X_1}^2 & \sigma_{X_1Y_1} & \sigma_{X_1Z_1} \\ \sigma_{X_1Y_1} & \sigma_{Y_1}^2 & \sigma_{Y_1Z_1} \\ \sigma_{X_1Z_1} & \sigma_{Y_1Z_1} & \sigma_{Z_1}^2 \end{bmatrix} & \Sigma_{12} = \begin{bmatrix} \sigma_{X_1X_2} & \sigma_{X_1Y_2} & \sigma_{X_1Z_2} \\ \sigma_{Y_1X_2} & \sigma_{Y_1Y_2} & \sigma_{Y_1Z_2} \\ \sigma_{Z_1X_2} & \sigma_{Z_1Y_2} & \sigma_{Z_1Z_2} \end{bmatrix} \\ \Sigma_{21} = \begin{bmatrix} \sigma_{X_1X_2} & \sigma_{Y_1X_2} & \sigma_{Z_1X_2} \\ \sigma_{X_1Y_2} & \sigma_{Y_1Y_2} & \sigma_{Z_1Y_2} \\ \sigma_{X_1Z_2} & \sigma_{Y_1Z_2} & \sigma_{Z_1Z_2} \end{bmatrix} & \Sigma_{22} = \begin{bmatrix} \sigma_{X_2}^2 & \sigma_{X_2Y_2} & \sigma_{X_2Z_2} \\ \sigma_{X_2Y_2} & \sigma_{Y_2}^2 & \sigma_{Y_2Z_2} \\ \sigma_{X_2Z_2} & \sigma_{Y_2Z_2} & \sigma_{Z_2}^2 \end{bmatrix} \end{bmatrix} \qquad (4)$$

Two equivalent methods can be used to compute the variance and standard deviation of the 3-D slope distance. First, matrix multiplication will provide the variance for the 3-D slope distance by using values of the partial derivatives from equation (3) and the covariance values from equation (4).

Second, the variance can be obtained by first substituting equations (3) and (4) into equation (1) and performing the algebraic operations. The result is:

$$\sigma^2 = \frac{\Delta X^2}{D^2}\left(\sigma_{X_1}^2 + \sigma_{X_2}^2\right) + \frac{\Delta Y^2}{D^2}\left(\sigma_{Y_1}^2 + \sigma_{Y_2}^2\right) + \frac{\Delta Z^2}{D^2}\left(\sigma_{Z_1}^2 + \sigma_{Z_2}^2\right)$$

$$+ \frac{2}{D^2}\left[\Delta X \Delta Y \left(\sigma_{X_1Y_1} + \sigma_{X_2Y_2} \right) + \Delta X \Delta Z \left(\sigma_{X_1Z_1} + \sigma_{X_2Z_2} \right) + \Delta Y \Delta Z \left(\sigma_{Y_1Z_1} + \sigma_{Y_2Z_2} \right) \right]$$

$$- \frac{2}{D^2}\left[\Delta X \Delta Y \left(\sigma_{X_1Y_2} + \sigma_{Y_1X_2} \right) + \Delta X \Delta Z \left(\sigma_{X_1Z_2} + \sigma_{Z_1X_2} \right) + \Delta Y \Delta Z \left(\sigma_{Y_1Z_2} + \sigma_{Z_1Y_2} \right) \right]$$

$$- \frac{2}{D^2}\left[\Delta X^2 \sigma_{X_1X_2} + \Delta Y^2 \sigma_{Y_1Y_2} + \Delta Z^2 \sigma_{Z_1Z_2} \right]. \qquad (5)$$

In either case, the standard deviation of the mark-to-mark (point-to-point) distance between any pair of points defined with geocentric X/Y/Z coordinates (in the same datum) and associated covariance matrices is commutative and computed as:

$$\sigma = \sqrt{\sigma^2} \tag{6}$$

Note here that the first two terms in equation (5) will provide the variance of the 3-D slope distance under the assumption of no correlation, i.e., the cross covariance values are all zero in the third and fourth terms of equation (5).

Data:

Short line data – Stations Pseudo and USPA from Chapter 11, Burkholder (2008):

Standpoint (Point 1): **Pseudo**

Geocentric coordinates	ECEF station covariance matrix, m^2 (Σ_{11} in eq. (4))		
X = -1,556,206.615 m	1.24245E-06	1.34309E-06	-9.21239E-07
Y = -5,169,400.740 m	1.34309E-06	4.50646E-06	-2.77505E-06
Z = 3,387,285.987 m	-9.21239E-07	-2.77505E-06	4.18236E-06

Forepoint (Point 2): **USPA**

Geocentric coordinates	ECEF station covariance matrix, m^2 (Σ_{22} in eq. (4))		
X = -1,555,678.579 m	2.16088E-06	2.34746E-06	-1.49586E-06
Y = -5,169,961.396 m	2.34746E-06	8.47410E-06	-5.01722E-06
Z = 3,386,700.089 m	-1.49586E-06	-5.01722E-06	6.81155E-06

Correlation Σ_{21} in equation (4), m^2			Correlation Σ_{12} in equation (4), m^2		
1.1405E-06	1.2934E-06	-8.8294E-07	1.1405E-06	1.2848E-06	-8.9269E-07
1.2848E-06	4.2401E-06	-2.5676E-06	1.2934E-06	4.2401E-06	-2.5759E-06
-8.9269E-07	-2.5759E-06	3.6030E-06	-8.8294E-07	-2.5676E-06	3.6030E-06

Medium line data – Stations FSRI and RASN in Southeastern Wisconsin – see Figure 1:

Standpoint (Point 1): **FRSI**

Geocentric coordinates	ECEF station covariance matrix, m^2 (Σ_{11} in eq. (4))		
X = 164,796.251 m	2.137380E-04	- 1.347237E-06	9.504950E-07
Y = -4,679,186.506 m	-1.347237E-06	2.7009126E-04	-4.442558E-05
Z = 4,316,889.723 m	9.504950E-07	-4.442558E-05	2.544635E-04

Forepoint (Point 2): **RASN**

Geocentric coordinates	ECEF station covariance matrix, m^2 (Σ_{22} in eq. (4))		
X = 153,059.630 m	2.136655E-04	-9.939590E-07	3.374190E-07
Y = -4,666,858.811 m	-9.939590E-07	2.702019E-04	-4.501482E-05
Z = 4,330,615.801 m	3.374190E-07	-4.501482E-05	2.551067E-04

Correlation Σ_{21} in equation (4), m^2			Correlation Σ_{12} in equation (4), m^2		
2.128074E-04	-5.581210E-07	9.322200E-08	2.128074E-04	-4.365860E-07	-7.018000E-09
-4.365860E-07	2.522824E-04	-3.039416E-05	-5.581210E-07	2.522824E-04	-3.054197E-05
-7.018000E-09	-3.054197E-06	2.401769E-04	9.322200E-08	-3.039416E-05	2.401769E-04

The long line data – Stations FSRI and SHAN in SE Wisconsin – see Figure 1:

Standpoint (Point 1): **FRSI**

Geocentric coordinates	ECEF station covariance matrix, m^2 (Σ_{11} in eq. (4))		
X = 164,796.251 m	2.137380E-04	-1.347237E-06	9.504950E-07
Y = -4,679,186.506 m	-1.347237E-06	2.700913E-04	-4.442558E-05
Z = 4,316,889.723 m	9.504950E-07	-4.442558E-05	2.544635E-04

Forepoint (Point 2): **SHAN**

Geocentric coordinates	ECEF station covariance matrix, m^2 (Σ_{22} in eq. (4))		
X = 182,409.481 m	2.130396E-04	-1.320212E-06	6.137860E-07
Y = -4,611,414.938 m	-1.320212E-06	2.498616E-06	-2.975560E-05
Z = 4,387,983.545 m	6.137860E-07	-2.975560E-05	2.390237E-04

Correlation Σ_{21} in equation (4), m^2			Correlation Σ_{12} in equation (4), m^2		
2.123000E-04	-3.206330E-07	-2.744470E-07	2.123000E-04	-6.70410E-08	-5.274200E-07
-6.70410E-08	2.3534120E-04	-1.924826E-05	-3.206330E-07	2.353420E-04	-1.911484E-05
-5.274200E-07	-1.911484E-05	2.288600E-04	-2.744470E-07	-1.924826E-05	2.288600E-04

Computation of 3-D Standard Deviations:

In each case, equation (5) is used to compute the standard deviation of the 3-D mark-to-mark distance between points for two separate circumstances:

- Using the full covariance matrix in equation (4) and
- Assuming no correlation between points by eliminating the third and fourth terms in equation (5).

For comparison, the slope of each vector is listed for verifying how close the slope distance is to being horizontal. The accuracy values shown below were computed as standard deviations (an unambiguous well-defined mathematical process) but are listed as network accuracy and local accuracy.

	Network Accuracy	Local Accuracy
Short line Pseudo to USPA - 0.968 km		
Slope = 89° 16’ 18”	0.00180 m	0.00105 m
Medium line FRSI to RASN – 21.87 km		
Slope = 89° 59’ 07”	0.02081 m	0.00179 m
Long line FRSI to SHAN – 99.78 km		
Slope = 89° 31’ 44”	0.02077 m	0.00251 m

Two important points to be made in reference to the slope distance standard deviations are:

1. These results are consistent with the conclusion in Soler/Han/Smith (2012) that significant improvement in the uncertainty between points is realized by using the full covariance matrix between points as opposed to using only the covariance matrices of the two endpoints. If the cross covariance matrices are not used (their values are zero) the two endpoints are statistically independent of each other.

2. The 3-D accuracy values above are readily computed and, because the lines are nearly horizontal, those values compare very favorably with the accuracy values in the following sections as computed using both Burkholder's (1999) one-rotation-matrix procedure and the rigorous two-rotation-matrix procedure defined by Soler/Smith (2010).

Computation of Local Accuracies:

This section shows computation of both network and local accuracies by Soler/Smith (2010) and Burkholder (1999) using the same data for short, medium, and long lines as was used for computing the standard deviation of the slope distance. First, a comparison is made between the values of the 3x3 covariance matrix in each case (short, medium, long) used in equation (14) by Soler/Smith (2010) and equation (11) by Burkholder (Discussion 2012) which is shown to be the same as the procedure used in Burkholder (1999). Following that, a comparison is made of the computed network and local accuracies of the distances between points.

For convenient reference, the equations for the local perspective covariance matrix of the separation between points are:

Soler/Smith (2010), eq. (14) $$\Sigma_{\Delta e,\Delta n,\Delta u} = R_1\Sigma_{11}R_1^T - R_1\Sigma_{12}R_2^T - R_2\Sigma_{21}R_1^T + R_2\Sigma_{22}R_2^T \quad (7)$$

Burkholder (2012), eq. (11) $$\Sigma_{\Delta e,\Delta n,\Delta u} = R\Sigma_{11}R^T - R\Sigma_{12}R^T - R\Sigma_{21}R^T + R\Sigma_{22}R^T \quad (8)$$

The 3x3 covariance matrix of the local perspective computed by equation (7) above for the <u>Short</u> line on the NMSU campus is:

$$\Sigma_{\Delta e,\Delta n,\Delta u} = \begin{bmatrix} 7.87895\text{E}-7 & 1.22998\text{E}-7 & 8.36203\text{E}-8 \\ 1.22998\text{E}-7 & 1.62944\text{E}-6 & 6.97400\text{E}-7 \\ 8.36203\text{E}-8 & 6.97400\text{E}-7 & 6.99166\text{E}-6 \end{bmatrix} \ \& \ \begin{bmatrix} \sigma_{\Delta e} = 0.0009\ m \\ \sigma_{\Delta n} = 0.0013\ m \\ \sigma_{\Delta u} = 0.0026\ m \end{bmatrix} \quad (9)$$

The 3x3 covariance matrix of the local perspective computed by equation (8) above for the <u>Short</u> line on the NMSU campus is:

$$\Sigma_{\Delta e,\Delta n,\Delta u} = \begin{bmatrix} 7.87895\text{E}-7 & 1.22991\text{E}-7 & 8.35829\text{E}-8 \\ 1.22991\text{E}-7 & 1.62948\text{E}-6 & 6.97427\text{E}-7 \\ 8.35829\text{E}-8 & 6.97427\text{E}-7 & 6.99162\text{E}-6 \end{bmatrix} \ \& \ \begin{bmatrix} \sigma_{\Delta e} = 0.0009\ m \\ \sigma_{\Delta n} = 0.0013\ m \\ \sigma_{\Delta u} = 0.0026\ m \end{bmatrix} \quad (10)$$

The 3x3 covariance matrix of the local perspective computed by equation (7) above for the <u>Medium</u> line in the GPS network in Wisconsin is:

$$\Sigma_{\Delta e,\Delta n,\Delta u} = \begin{bmatrix} 1.73686\text{E}-6 & 4.01300\text{E}-8 & 2.96738\text{E}-7 \\ 4.01300\text{E}-8 & 3.81055\text{E}-6 & -1.23848\text{E}-6 \\ 2.96738\text{E}-7 & -1.23848\text{E}-6 & 6.11934\text{E}-5 \end{bmatrix} \ \& \ \begin{bmatrix} \sigma_{\Delta e} = 0.0013\ m \\ \sigma_{\Delta n} = 0.0020\ m \\ \sigma_{\Delta u} = 0.0078\ m \end{bmatrix} \quad (11)$$

The 3x3 covariance matrix of the local perspective computed by equation (8) above for the Medium line in the GPS network in Wisconsin is:

$$\Sigma_{\Delta e,\Delta n,\Delta u} = \begin{bmatrix} 1.73701E-6 & 3.68349E-8 & 3.51352E-7 \\ 3.68349E-8 & 3.81215E-6 & -1.32017E-6 \\ 3.51352E-7 & -1.32017E-6 & 6.11844E-5 \end{bmatrix} \quad \& \quad \begin{bmatrix} \sigma_{\Delta e} = 0.0013\ m \\ \sigma_{\Delta n} = 0.0020\ m \\ \sigma_{\Delta u} = 0.0078\ m \end{bmatrix} \qquad (12)$$

The 3x3 covariance matrix of the local perspective computed by equation (7) above for the Long line in the GPS network in Wisconsin is:

$$\Sigma_{\Delta e,\Delta n,\Delta u} = \begin{bmatrix} 2.07569E-6 & 3.98260E-7 & 1.13848E-6 \\ 3.98260E-7 & 6.39059E-6 & -4.44871E-6 \\ 1.13848E-6 & -4.44871E-6 & 7.88627E-5 \end{bmatrix} \quad \& \quad \begin{bmatrix} \sigma_{\Delta e} = 0.0014\ m \\ \sigma_{\Delta n} = 0.0025\ m \\ \sigma_{\Delta u} = 0.0089\ m \end{bmatrix} \qquad (13)$$

The 3x3 covariance matrix of the local perspective computed by equation (8) above for the Long line in the GPS network in Wisconsin is:

$$\Sigma_{\Delta e,\Delta n,\Delta u} = \begin{bmatrix} 2.07098E-6 & 3.99797E-7 & 9.54143E-7 \\ 3.997967E-7 & 6.42459E-6 & -5.22940E-6 \\ 9.54143E-7 & -5.22940E-6 & 7.87179E-5 \end{bmatrix} \quad \& \quad \begin{bmatrix} \sigma_{\Delta e} = 0.0014\ m \\ \sigma_{\Delta n} = 0.0025\ m \\ \sigma_{\Delta u} = 0.0089\ m \end{bmatrix} \qquad (14)$$

Comparison of the covariance matrices computed using equation (7) and equation (8) for all three (short, medium, long) lines shows that they are, in fact, slightly different. However, when taking the square root of the diagonal elements (down to the tenth of a millimeter), there is no difference between the results obtained from equation (7), called rigorous local accuracy by Soler/Smith, and equation (8), computed as a standard deviation of the horizontal distance by Burkholder (2008) and called local accuracy.

In order to facilitate a comparison of similar quantities, the covariance values in equations (9) & (10), (11) & (12), and (13) & (14) were used to compute the standard deviation of the horizontal distance - the assumption in Burkholder (1999) - and the standard deviation of the separation between points - called rigorous local accuracy by Soler/Smith (2010).

As listed in Burkholder (1999) in equations (12) to (15), the standard deviation of the horizontal distance is computed using the partial derivatives of equation (15) below and the local 3x3 covariance matrix computed above.

$$HD = \sqrt{\Delta e^2 + \Delta n^2} \qquad (15)$$

The partial derivatives in the Jacobian matrix as obtained from equation (15) are:

$$J = \begin{bmatrix} \frac{\partial HD}{\partial \Delta e} & \frac{\partial HD}{\partial \Delta n} & \frac{\partial HD}{\partial \Delta u} \end{bmatrix} = \begin{bmatrix} \frac{\Delta e}{HD} & \frac{\Delta n}{HD} & 0 \end{bmatrix} \qquad (16)$$

The matrix product defined in equation (1) was used to multiply the Jacobian values of equation (16) and the local 3x3 covariance matrices in equations (10), (12), and (14) to compute the variances (and standard deviations) of the horizontal distance for the short, medium, and long lines as follows:

	Network Accuracy	Local Accuracy
Short line Pseudo to USPA - 0.968 km	0.00180 m	0.00105 m
Medium line FRSI to RASN – 21.866 km	0.02081 m	0.00179 m

Long line FRSI to SHAN – 99.784 km	0.02077 m	0.00250 m

Since the derivation of local accuracy by Soler/Smith (2010) does not specify horizontal distance, the local accuracy computation here uses all three components of the separation between points

$$SD = \sqrt{\Delta e^2 + \Delta n^2 + \Delta u^2} \tag{17}$$

The partial derivatives in the Jacobian matrix as obtained from equation (17) are:

$$J = \begin{bmatrix} \frac{\partial SD}{\partial \Delta e} & \frac{\partial SD}{\partial \Delta n} & \frac{\partial SD}{\partial \Delta u} \end{bmatrix} = \begin{bmatrix} \frac{\Delta e}{SD} & \frac{\Delta n}{SD} & \frac{\Delta u}{SD} \end{bmatrix} \tag{18}$$

The matrix product defined in equation (1) was used to multiply the Jacobian values of equation (18) and the local 3x3 covariance matrices in equations (9), (11), and (13) to compute the variances (and standard deviations) of for the short, medium, and longs lines as follows (Soler/Smith 2010):

	Network Accuracy	Local Accuracy
Short line Pseudo to USPA - 0.968 km	0.00180 m	0.00105 m
Medium line FRSI to RASN – 21.866 km	0.02081 m	0.00179 m
Long line FRSI to SHAN – 99.784 km	0.02078 m	0.00254 m

Summary of results:

A summary of results, both network and local, for all three lines (short, medium, long) and three methods (3-D standard deviation, Burkholder 1 rotation matrix, and Soler/Smith 2 matrix rotations) is included in the following table. All units are meters. Showing five decimal places of meters cannot be justified except to show where differences in the answers begin to occur.

	3-D Slope Distance		Burkholder, Hor. Dist.		Soler/Smith	
Line	Network	Local	Network	Local	Network	Local
Short	0.00180 m	0.00105 m	0.00180 m	0.00105 m	0.00180 m	0.00105 m
Medium	0.02081 m	0.00179 m	0.02081 m	0.00179 m	0.02081 m	0.00179 m
Long	0.02077 m	0.00251 m	0.02077 m	0.00250 m	0.02078 m	0.00254 m

Observations/Conclusions:

1. All three methods listed herein give essentially the same answer. Each approach has merit.

2. The material in Burkholder (2008) remains valid. That derivation is rigorous for computation of standard deviation of the horizontal distance between points. It appears that the definition horizontal distance over a long distance becomes meaningless before the computation of its standard deviation suffers. See Burkholder (1991).

3. The method of Soler/Smith is not restricted to horizontal distance and there is no limit on the distance between points. As such, it enjoys the commutative property in that it provides the same answer of local accuracy for a pair of points whether going from Point 1 to Point 2 or from Point 2 to Point 1.

4. The computation of standard deviation for a 3-D spatial distance also has no limit on the distance between points and enjoys the commutative property. The attractive feature of this method is that the answer is computed directly in terms of the 3-D inverse distance between points and the full covariance matrix of that pair of points. No rotation matrices are required and there are no approximations in the algorithm.

Further Study:

Considerations in this paper focus on computation of standard deviation (or network/local accuracy) based upon a given pair of points and their covariance matrices. The computed relative position of one point with respect to another can be different if different covariance values are used. The covariance values for points in the network may be quite different depending upon which control points are held in an adjustment (how the adjustment was constrained). For example, the Wisconsin GPS network cited in this example was minimally constrained with station "FOLA" being assigned 0.010 m standard deviation in all three components. Other scenarios were also computed and the results were found to support and be consistent with the "dam monitoring" example described in Burkholder (2004). The study and comparison of those "local accuracies" will be reported in a separate paper.

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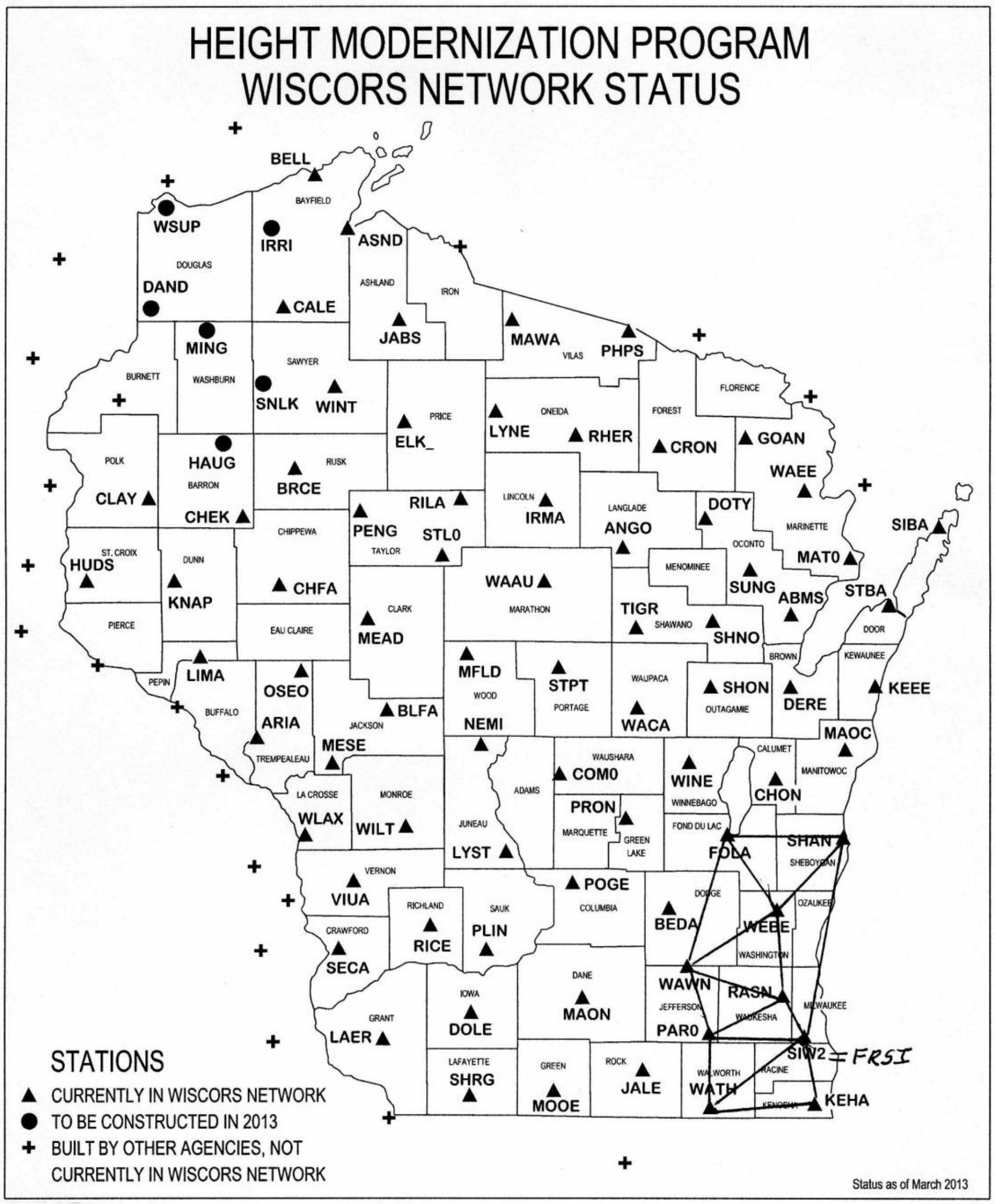

Base Map – WISDOT, 2013
Network Overlay – Burkholder, 2013

Figure 1 - GPS Network in SE Wisconsin