Comparison of geodetic, state plane, and geocentric computational models

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Abstract:
Surveyors make measurements and process those data to compute positions. Plane surveying uses two-dimensional (2-D) coordinates based on assumptions of a flat-earth. Plats, maps, and land boundary descriptions are prepared consistent with those assumptions and methods. The client is well served and, over the years, many "local practice" businesses have functioned successfully. But, as technology advances and as the scope of a project or service area gets larger, those flat earth assumptions become limiting and plane surveyors are exposed to new challenges. Modern measurement systems evolved during the digital revolution and now routinely collect three-dimensional (3-D) digital geospatial data. Likewise, computational processes now used in data reduction go well beyond the flat earth assumptions. And, models for processing 3-D digital geospatial data have evolved from flat earth models to various ellipsoidal models to a plethora of map projections to 3-D models that support computations in 3-D space worldwide – e.g., the global spatial data model (GSDM). This paper includes a comparison of three models used to determine a 3-D geodetic position based upon a simple total-station sideshot from a known station. The three methods are geodetic on the ellipsoid, state plane on a mapping grid, and geocentric in 3-D space.

Introduction:

Station "Reilly" (PID AI5445) is geodetic control monument on the campus of New Mexico State University (NMSU) in Las Cruces. Used extensively by NMSU surveying engineering students, station "Reilly" was established as part of the New Mexico high-accuracy-reference-network (HARN) and its position was published by the National Geodetic Survey (NGS) on the North American Datum of 1983 (1992) - NAD 83 (1992). Although no longer referred to as a HARN station, NGS has subsequently published the position of "Reilly" on NAD 83 (2007) and NAD 83 (2011). As a learning exercise, students occupied "Reilly" with a total-station surveying instrument, backsighted a known azimuth mark, turned the horizontal angle, and observed the slope distance to the target – a retro-reflector sitting on the desk of the NMSU Associate Dean of Engineering. The height of the instrument (HI), height of target (HT), and zenith direction to the target were also observed. This paper uses those data in three different models (geodetic, state plane, and geocentric) to compute an un-monumented 3-D geodetic position on the Associate Dean's desk. It appears that, with no loss of integrity, the geocentric model has advantages of simplicity not shared by the other two models.

Models and Objective:

Models provide a conceptual connection between the abstract and human experience. Some models – for example the flat-earth model - are simple and easy to use. But spatial data models become more complex as needed to preserve the integrity of survey measurements and to account for geometrical relationships that extend beyond a local perspective. A general statement is, the "best" model is the simplest one that does not sacrifice geometrical integrity. Therefore, selection of the most appropriate spatial data model for a given application often involves a balance between simplicity and integrity. The objective of this paper is to compare ease-of-use and the complexity of three computational procedures (models) that provide essentially identical answers for a position on the top of the NMSU Engineering Associate Dean's desk:

- Traditional geodetic computations on the ellipsoid.
- NM Central Zone state plane coordinates.
- Geocentric X/Y/Z values computed in 3-D space.

Background, Control Values, and Measurements (Common to all three models):

The office of the Associate Dean of Engineering at NMSU is on the ground floor of Goddard Hall on the NMSU campus. Station "Reilly" is a ground level brass tablet set in the top/middle of a massive concrete vault in an open area next to Goddard Hall. The 2015 NGS data sheet lists the geodetic latitude and longitude position, state plane values, and geocentric earth-centered earth-fixed (ECEF) coordinates for station "Reilly." An approximate geoid height at "Reilly" is also listed on the data sheet. For this comparison, the North American Vertical Datum 1988 (NAVD 88) elevation for station "Reilly" was determined from local first-order benchmarks using GPS and geoid modeling – see http://www.globalcogo.com/ReilElev.pdf - and a finial on Skeen Hall about 240 meters westerly of station "Reilly" was sighted for azimuth orientation. The azimuth to the finial was computed from 4 sets of Wild T-2 Polaris observations in 2001. A Laplace correction obtained using the NGS program "Deflect99" was used to compute a geodetic azimuth from the observed astronomic azimuth.

The following NAD 83 (2011) values were taken from the NGS data sheet.

	Geodetic	State Plane	Geocentric
Station	ϕ = 32° 16' 55."93001 N	E = 452,506.490 m	X = -1,556,177.595 m
"Reilly"	λ = 106° 45' 15."16035 W	N = 142,268.771 m	Y = -5,169,235.284 m
	= 253° 14' 44."83965 E		Z = 3,387,551.720 m
	Ellipsoid height		= 1,166.543 m
	Geoid height (Geoid 12B)		= -23.94 m
	Grid scale factor		= 0.99992781
	Convergence		= -0° 16' 09."5

Other values used in the computations include:

GRS80 ellipsoid parameters:	a = 6,378,137.000m and e^2 = 0.006694380023
Seconds per radian	spr = 206,264.806247
NAVD 88 elevation of "Reilly"	H = 1,190.497 m
Geodetic azimuth from "Reilly" to finial on Skeen Hall	α_{BS} = 272° 11' 09"

Measurements:

EDM slope distance (corrected for temperature & prism off-set)	= 78.452 m
Angle right from finial to reflector on desk (mean of 4 sets D/R)	= 269° 23' 08"
Zenith direction to center of reflector (mean of 2 sets D/R)	= 090° 54' 08"
Height of instrument at "Reilly" (HI)	= 1.682 m
Height of target on desk (HT)	= 0.366 m

Equations and Computations:

1. Geodesy computations on the ellipsoid:
   Equations for forward (also called "direct") geodetic computations on the ellipsoid are given in sources such as Vincenty (1975 and 1980) and Jank and Kivioja (1980). The equations used here are for one element of a geodetic line as described by Burkholder (2008) and based on the numerical integration method used by Jank and Kivioja (1980) who claim that millimeter accuracy of a computed position is maintained half way around the world when the individual line element used in the numerical integration is 200 meters or less. Longer elements can be used on shorter lines while maintaining the same millimeter accuracy. Burkholder (2008) also describes a test for checking and assuring the accuracy of a geodetic forward computation.

$$\varphi_{desk} = \varphi_{Reilly} + \Delta\varphi \qquad (1a)$$

$$\lambda_{desk} = \lambda_{Reilly} + \Delta\lambda \qquad \text{See step-by-step equations below.} \qquad (1b)$$

$$H_{desk} = H_{Reilly} + \Delta H \qquad (1c)$$

$$\Delta\varphi'' = \frac{S \cos \alpha_{Geo}}{M}\ spr \qquad (2)$$

$$\Delta\lambda'' = \frac{S \sin \alpha_{Geo}}{N \cos \varphi}\ spr \qquad (3)$$

$$\Delta H = HI + SD \cos Z - HT + (curvature\ and\ refraction) \qquad (4)$$

$$S = ellipsoidal\ distance = SD * \sin Z * \frac{R_m}{R_m + h} \qquad (5)$$

SD = Slope distance

$$\alpha_{Geo} = geodetic\ azimuth = \alpha_{BS} + angle\ right \qquad (6)$$

Z = zenith direction to target (mean of 2 D/R sets)

HI = height of instrument

HT = height of target

$$M = \frac{a(1-e^2)}{(1-e^2\ sin^2\varphi)^{1.5}} \qquad \text{Radius of curvature in the Meridian} \qquad (7)$$

$$N = \frac{a}{\sqrt{1-e^2\ sin^2\varphi}} \qquad \text{Radius of curvature in the Prime Vertical} \qquad (8)$$

$$R_m = \sqrt{M * N} \qquad \text{Gaussian mean radius} \qquad (9)$$

Computations (use latitude and east longitude at station Reilly):

$$M = \frac{6{,}378{,}137.00*(1-0.006694380023)}{(1-0.006694380023*sin^2(32°\ 16'\ 55.''93001))^{1.5}} = 6{,}353{,}629.826\ \text{m}$$

$$N = \frac{6{,}378{,}137.000}{\sqrt{1-0.006694380023*sin^2(32°\ 16'\ 55.''93001)}} = 6{,}384{,}235.531\ \text{m}$$

$$R_m = \sqrt{6{,}353{,}629.826 * 6{,}384{,}235.532} = 6{,}368{,}914.294\ \text{m}$$

$h = ellipsoid\ height\ at\ Reilly$, <u>from NGS data sheet</u> = 1,166.543 m

$$S = 78.452 * \sin(90° \, 54'08") * \frac{6{,}368{,}914.294}{6{,}370{,}080.837} \quad = \quad 78.428 \text{ m}$$

$$\alpha_{Geo} = 272° \, 11'09" + 269° \, 23' \, 08 - 360° \quad = \quad 181° \, 34' \, 17"$$

$$\Delta\varphi" = \frac{78.428 * \cos(181° \, 34'17")}{6{,}353{,}629.826} * spr \quad = \quad -2."54514$$

$$\Delta\lambda" = \frac{78.428 * \sin(181° \, 34'17")}{6{,}384{,}235.531 * \cos(32° \, 16'55."93001)} * spr \quad \text{(East } \Delta\lambda\text{)} \quad = \quad -0."08219$$

$$\Delta H = 1.682 + 78.452 * \cos(90° \, 54'08") - 0.366 \quad = \quad 0.081 \text{ m}$$

(In this case, curvature and refraction is < 0.0005 m and is ignored.)

And, the results are:

$$\varphi_{desk} = 32° \, 16'55."93001 - 2."54514 \quad = \quad 32° \, 16' \, 53."38487 \text{ N}$$

$$\lambda_{desk} = 253° \, 14'44."83965 - 0."08219 \quad = 253° \, 14' \, 44."75746 \text{ E}$$

$$= 106° \, 45' \, 15."24254 \text{ W}$$

$$H_{desk} = 1{,}190.497 m + 0.081 \, m = \quad = 1{,}190.578 \text{ m}$$

2. State Plane Coordinate Computations – NM Central Zone:

State plane coordinates are computed for the top of the desk and geodetic positions are computed from the state plane values. Although the equations are not listed herein, the geodetic latitude and longitude are computed from state plane coordinates using the algorithm given in Stem (1989). Elevation on the desk is computed the same way as used in the geodetic method. Computation of state plane coordinates is:

$\varphi_{desk} =$ Computed from NM Central Zone state plane coordinates. (10a)

$\lambda_{desk} =$ Computed from NM Central Zone state plane coordinates. (10b)

$H_{desk} = H_{Reilly} + \Delta H$ (Same as in previous method.) (1c)

$$E_{desk} = E_{Reilly} + HD_{Grid} * sin(Az_{Grid}) \quad (11)$$

$$N_{desk} = N_{Reilly} + HD_{Grid} * \cos(Az_{Grid}) \quad (12)$$

Where:

$$HD_{Grid} = SD * \sin(Zenith \; Direction) * combined \; factor \quad (13)$$

$$combined \; factor = grid \; scale \; factor * elevation \; factor \quad (14)$$

$$elevation \; factor = \frac{R_m}{R_m + h} \quad (15)$$

$$Az_{Grid} = Az_{Grid \; to \; BS} + angle \; right \quad (16)$$

$$Az_{Grid \; to \; BS} = \alpha_{Geo \; to \; BS} - convergence \; at \; "Reilly" \quad (17)$$

Computations:

$$Az_{Grid \; to \; BS} = 272° \, 11'09" - (-00° \, 16' \, 09.5) \quad = \quad 272° \, 27' \, 19"$$

$$Az_{Grid \; to \; desk} = 272° \, 27'19" + 269° \, 23' \, 08 - 360° \quad = \quad 181° \, 50' \, 27"$$

$$combined \; factor = 0.99992781 * \frac{6{,}368{,}914.294}{6{,}370{,}080.837} \quad = \quad 0.999744695$$

$$HD_{Grid} = 78.452 * \sin(90° \, 54'08") * 0.999744695 \quad = \quad 78.422 \text{ m}$$

$Easting_{desk} = 452{,}506.490 + 78.422 * \sin(181°\ 50'26."6)$ = 452,503.971 m

$Northing_{desk} = 142{,}268.771 + 78.422 * \cos(181°\ 50'26."6)$ = 142,190.389 m

$H_{desk} = 1{,}190.497\ m + 0.081\ m$ (same as before) = 1,190.578 m

Geodetic latitude and longitude for the New Mexico Central Zone NAD83 state plane coordinates give:

Latitude on Dean's desk = 32° 16' 53."38488 N

Longitude on Dean's desk = 106° 45' 15."24253 W

Elevation on Dean's desk = 1,190.578 m

3. Geocentric earth-centered earth-fixed (ECEF) X/Y/Z coordinates:
Equations for computing ECEF geocentric coordinates are found in Chapter 1 of Burkholder (2008) which describes the global spatial data model (GSDM). The equations and procedures can also be found in other geodesy texts. When using the GSDM for geodetic computations, the computations are performed in 3-D space to obtain the geocentric X/Y/Z coordinate values. For purposes of comparison with the other two methods, the geocentric X/Y/Z coordinates need to be converted to geodetic latitude, longitude, and ellipsoid height. Geoid heights are required to determine orthometric heights (elevation) from ellipsoid heights. The NGS program, Geoid12B, was used to compute geoid heights and those geoid heights were used to compute the NAVD 88 elevation on the desk.

$\varphi_{desk} =$ Computed from X/Y/Z geocentric ECEF coordinate values. (18a)

$\lambda_{desk} =$ Computed from X/Y/Z geocentric ECEF coordinate values. (18b)

$H_{desk} = H_{Reilly} + \Delta H$ <u>Different than equation (1c)</u> (18c)

$$X_{desk} = X_{Reilly} + \Delta X \quad (19)$$

$$Y_{desk} = Y_{Reilly} + \Delta Y \quad (20)$$

$$Z_{desk} = Z_{Reilly} + \Delta Z \quad (21)$$

$$\Delta X = -\Delta e \sin\lambda - \Delta n \sin\varphi \cos\lambda + \Delta u \cos\varphi \cos\lambda \quad (22)$$

$$\Delta Y = \Delta e \cos\lambda - \Delta n \sin\varphi \sin\lambda + \Delta u \cos\varphi \sin\lambda \quad (23)$$

$$\Delta Z = \Delta n \cos\varphi + \Delta u \sin\varphi \quad (24)$$

Note: Equations (22) to (24) use north latitude & east longitude at Station "Reilly."

$$\Delta e = SD \sin Z \sin \alpha_{Geo} \quad (25)$$

$$\Delta n = SD \sin Z \cos \alpha_{Geo} \quad (26)$$

$$\Delta u = SD \cos Z + HI - HT \quad (27)$$

As before:

$SD = slope\ distance$

$Z = zenith\ direction\ to\ reflector$

$\alpha_{Geo} = azimuth\ to\ reflector\ on\ desk = \alpha_{BS} + angle\ right$ (6)

Computations:

$\alpha_{Geo} = 272°\ 11'09" + 269°\ 23'\ 08" - 360°$ = 181° 34' 17"

$\Delta e = 78.452 \sin(90°\ 54'08) * \sin(181°\ 34'\ 17")$ = -2.151 m

$\Delta n = 78.452 \sin(90°\ 54'08) * \cos(181°\ 34'\ 17")$ = -78.413 m

$\Delta u = 78.452 \cos(90°\ 54'\ 08") + 1.682 - 0.366$ = 0.081 m

ΔX = -(-2.151) sin(253°14’ 44.”83965) – (-78.413) sin (32°16’ 55.”93001) * cos (253° 14’ 44.”83965) + 0.081 cos(32° 16’ 55.”93001) * cos(253° 14’ 44.”83965) = -14.152 m

ΔY = (-2.151) cos(253°14’ 44.”83965) – (-78.413) sin (32°16’ 55.”93001) * sin (253° 14’ 44.”83965) + 0.081 cos(32° 16’ 55.”93001) * sin(253° 14’ 44.”83965) = -39.547 m

$$\Delta Z = -78.413 \cos(32° \, 16' \, 55.''93001) + 0.081 \sin(32° \, 16' \, 55.''93001)$$
= -66.249 m

$X_{desk} = -1{,}556{,}177.595 + (-14.152)$ = -1,556,191.747 m
$Y_{desk} = -5{,}169{,}235.284 + (-39.547)$ = -5,169,274.831 m
$Z_{desk} = 3{,}387{,}551.720 + (-66.249)$ = 3,387,485.471 m

These geocentric X/Y/Z values need to be converted to latitude, longitude, and ellipsoid height (and to elevation) for a comparison to be made. The longitude computation is very straight forward but the latitude and ellipsoid height computations are more challenging. Techniques such as one of those described in Meyer (2010) can be used with excellent results. But the iteration method used here provides fully rigorous results with fewer mathematical gymnastics.

Since the geocentric X and Y values are both negative, the east longitude lies in the third quadrant of the equator and is computed (with due regard to radian units) as:

$\lambda = 180° + atan\left(\frac{Y}{X}\right)$ = East longitude (28)

Longitude = $180° + atan\left(\frac{-5{,}169{,}274.831}{-1{,}556{,}191.747}\right)$ = 253° 14’ 44.”75745 E
= 106° 45’ 15.”24255 W

Closed form equations for computing geocentric X/Y/Z coordinates from latitude, longitude, and ellipsoid height are called a BK1 transformation in Burkholder (2008) and given as:

$$X = (N + h) \cos\varphi \cos\lambda \quad (29)$$
$$Y = (N + h) \cos\varphi \sin\lambda \quad (30)$$
$$Z = [N(1 - e^2) + h] \sin\varphi \quad (31)$$

Where: N = Radius of curvature in Prime Vertical (8)
h = Ellipsoid height

A mathematical inversion of equations (29), (30), and (31) can be used to compute the latitude and ellipsoid height from the X/Y/Z coordinates. That inversion is also closed form but must be solved using iteration. Solving those inverted equations is referred to as a BK2 transformation and given in Chapter 6 Burkholder (2008) as:

$P = \sqrt{X^2 + Y^2}$, an intermediate value (32)

$\varphi_0 = \arctan\left(\frac{Z}{P\,(1 - e^2)}\right)$, “seed” value for subsequent use. (33)

$N_0 = \frac{a}{\sqrt{1 - e^2 \sin^2\varphi_0}}$, needed in next step (34)

$\varphi_i = \arctan\left[\frac{Z}{P}\left(1 + \frac{e^2 N_0 \sin\varphi_0}{Z}\right)\right]$, second & subsequent values (35)

$$N_i = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_i}}, \text{ second and subsequent values} \quad (36)$$

Once the solution for latitude has converged sufficiently, the ellipsoid height is computed using the latest values of ø and N:

$$h = \frac{P}{\cos\varphi} - N \quad (37)$$

Using a spreadsheet, the latitude for the top of the Dean's desk is computed as:

Iteration	Latitude (rad)	Difference	Normal	Difference
0	0.563418945242		6,384,570.81481 m	
1	0.563418550755	-0.000000394487	6,384,235.29594 m	-335.51888 m
2	0.563418390041	-0.000000160715	6,384,235.29283 m	-0.00311 m
3	0.563418389270	-0.000000000770	6,384,235.29281 m	-0.00002 m
4	0.563418389267	-0.000000000004	6,384,235.29281 m	-0.00000 m

The latitude is computed by converting radians to degrees-minutes-seconds using the conversion *spr* = 206,264.806247096 seconds/radian.

ø = 0.563418389267 * 206,264.806247 = 116,213."384899 = 32° 16' 53."38490

Twelve decimal places of latitude or longitude in radians translates to 6 decimal places of seconds when expressed as degrees, minutes, and seconds. More decimal places are included in the latitude tabulation than can be justified. This is done to show where differences begin to occur. It is safer to use more iterations than needed than to stop the iteration prematurely. Good judgment is essential in reporting and interpreting results. In this case, the comparison between models is made at 5 decimal places of seconds for latitude and longitude (0."00001 ≅ 0.0003 m) but, due to original observations being limited to the millimeter, the computed position can only be justified at 5 decimal places of seconds.

To compute the elevation (orthometric height) of the top of the Dean's desk, the ellipsoid height must be converted to elevation. Milbert (1991) states that modeled geoid height differences are more accurate than a modeled geoid height at a single point. Therefore, two alternates are included for computing the NAVD 88 elevation of the top of the desk:

1. Apply the modeled geoid height at the desk as obtained from the NGS Geoid 12B model. This method relies on the absolute value of modeled geoid height.

$$H_{desk} = h_{desk} - geoid\ height_{desk} \quad (38)$$

2. Determine the geoid height at both station "Reilly" and on the desk using the NGS Geoid 12B model. The difference in geoid height combined with the difference in ellipsoid height should provide a stronger solution than using only a single modeled geoid height.

$$H_{desk} = H_{Reilly} + (h_{desk} - h_{Reilly}) - (geoid\ height_{desk} - geoid\ height_{Reilly}) \quad (39)$$

Using NGS interactive software, Geoid12B computations of geoid heights at Station "Reilly" and on the Dean's desk are:

Geoid height at Station "Reilly"	-23.943 m
Geoid height on Dean's desk	-23.944 m

Elevation on Dean's desk using the alternative 1:

$H_{desk} = 1{,}166.624\ m - (-23.944\ m) =$ 1,190.568 m

Elevation on Dean's desk using alternative 2:

$H_{desk} = 1{,}190.497 + (1{,}166.624 - \ 1{,}166.543) - (-23.944 - (-)23.943) =$ 1,190.579 m

Summary of Results:

A comparison of geodetic position and elevation of the top of the Dean's desk for all three methods is:

	Geodetic	State Plane	3-D Geocentric
Latitude	32° 16' 53."38487 N	32° 16' 53."38488 N	32° 16' 53."38490 N
Longitude	106° 45' 15."24254 W	106° 45' 15."24253 W	106° 45' 15."24255 W
Elevation	1,190.578 m	1,190.578 m	1,190.568 m (alt. 1)
			1,190.579 m (alt. 2)

Conclusions and Comments:

1. All three methods yielded latitude/longitude values within 0.00003 seconds of arc – that is agreement within about 0.001 meter – consistent with the quality of the observations.

2. Elevations derived from the geodetic and state plane methods are identical. There are two geocentric solution elevations. The first alternative uses the modeled absolute geoid model value for the top of the desk and the result agrees with other methods within 0.011 meters. The second alternative uses ellipsoid height difference along with modeled geoid height difference and the result agrees with the first two methods within 1 millimeter. That illustrates the importance of using geoid modeled differences (relative) as opposed to using absolute geoid heights.

3. The geodetic model uses differential geometry equations on the ellipsoid. Although those geodesy equations are straight-forward, they can be intimidating to persons not familiar with same. But, all data and equations are listed herein.

4. Except maybe for needing to use grid azimuth and grid distance, Equations (11) and (12) in the state plane model are quite familiar to plane surveyors. Equations (13) to (17) deal with concepts of grid scale factors, elevation factors, combined factors, and convergence. Although not needed or used in the 3-D model, they are required when using the state plane model.

5. The process of computing latitude and longitude from state plane coordinates is rather complicated and the equations are not listed. However, those computations have become ingrained in modern practice and software is readily available for making those conversions. Stem (1989) is an excellent source for equations and algorithms for NAD 83 state plane coordinate conversions.

6. The geocentric computations are performed in 3-D space using rules of solid geometry. The equations for geocentric computations are readily available in Burkholder (2008) and other sources. Additional on-line resources are available as follows:

www.globalcogo.com/GM008.pdf	Gives equations for ø/λ/h to X/Y/Z (BK1)
www.globalcogo.com/GM009.pdf	Gives equations for X/Y/Z to ø/λ/h (BK2)
www.globalcogo.com/GM010.pdf	Diagrams illustrating BK1 & BK2 comps.
www.globalcogo.com/DD-BK2.xlsx	Excel file for Deans Desk BK2 comps.
www.globalcogo.com/DD-BK2.pdf	PDF file of Deans Desk BK2 comps.

7. The BK2 computation is the most difficult part of the GSDM geometrical computations. The BK1 transformations from latitude/longitude/height to geocentric X/Y/Z coordinates are fairly straight forward but the reverse process (BK2) inverts those equations. The solution is also closed form, but these equations must be iterated for a solution. Iteration is used in the above spread sheet file for BK2 computations.

8. Other alternatives to the iteration procedure used in the BK2 spreadsheet include:
    - T. Vincenty (1980) devised a non-iterative algorithm as used in the following link - www.globalcogo.com/GM009.pdf . Comparisons have been made using these equations with excellent results – even out to satellite heights.
    - Equations (28) to (35) of Meyer (2010) can be used for a reliable non-iterative computational alternative.

9. This author concludes that the 3-D GSDM can be used to perform most 3-D spatial data computations in 3-D space with greater ease and efficiency than performing similar computations on the ellipsoid or using a map projection such as state plane or UTM coordinates. Furthermore, there need be no sacrifice of geometrical integrity when using the GSDM.

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