Coordinates, Calculators, and Intersections

by Earl F. Burkholder

Abstract. Programmable calculators have become quite indispensable to anyone performing surveying calculations. Trigonometric formulas used in plane coordinate computations are universally understood and many have programmed them for various calculators; some efficiently and correctly, others not so. This paper presents formulas and calculator procedures for coordinate geometry and intersection computations which are superior in accuracy and efficiency to those appearing in recent surveying texts. Greater accuracy is obtained by utilizing coordinate differences in the intersection formulas. Greater efficiency is achieved through use of polar-rectangular conversions and by exploiting similarities found in the solutions of various intersection problems.

Introduction

Programmable calculators have become an indispenable tool for anyone performing surveying calculations. Although tedium of looking up trigonometric functions and recording numerous intermediate values has been eliminated, performing computations efficiently is still desirable. Additionally, the professional surveyor is responsible for correctness of the result and should know what a "canned" program is doing with the data. This paper presents formulas for coordinate geometry computations which are superior in accuracy and efficiency to many being used. Greater accuracy is obtained by using coordinate differences rather than the entire coordinate value (i.e., state plane coordinates) in the intersection formulas. Greater efficiency is achieved through use of the "surveyor's reference system" in the polar-rectangular conversions and by exploiting similarities found in various intersection problems.

Goal

The goal here is to present rigorous, efficient calculator and programming procedures for the following computations:

- Forward (Traverse)
- Inverse
- Line-line intersection (bearing-bearing)
- Line-circle intersection (bearing-distance)
- Circle-circle intersection (distance-distance)
- Perpendicular offset

It is possible to program each problem the way it would be solved longhand. However, it is more efficient to use built-in functions for the Forward and Inverse and to solve the intersections symbolically before programming them.

Definitions and Conventions

Although redundant for most, definitions and conventions to be followed are stated specifically. There must be no ambiguity in the programmer's mind or the user's understanding as to the meaning or use of any element in the solution of a problem. A computer does only and exactly what it is told to do.

Surveyor's Reference System: A two-dimensional plane cartesian coordinate system is used for surveying computations and includes:

- A set of mutually perpendicular axes consisting of:

  a. The abscissa, a horizontal line along which the X distance is measured and,

  b. The ordinate, a vertical line along which the Y distance is measured.

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• Labeling and use of map directions as follows:

a. North, the positive Y axis direction.

b. East, the positive X axis direction.

c. South, the negative Y axis direction.

d. West, the negative Y axis direction.

• Use of North as the reference direction, 000°00′00″.

• A positive clockwise rotation measured in degrees, minutes, and seconds from 0° to 360° (azimuths).

• Quadrant labeling as:

a. Northeast, Quadrant I

b. Southeast, Quadrant II

c. Southwest, Quadrant III

d. Northwest, Quadrant IV

Math/Science Reference System: Practically all calculators are built or "hardwired" conventionally as follows:

• The trigonometric functions normally operate in decimal degrees. Radians or grads can be specified.

• The polar/rectangular conversions are based upon the math/science coordinate system. It is the same as the surveyor reference system except:

a. No map directions are used.

b. The reference direction is along the X axis.

c. Positive rotation is counterclockwise.

d. Quadrants are labeled counterclockwise (Fig. 1).

Each reader is responsible to reconcile the differences between the coordinate system hardwired into the particular calculator and that used for surveying computations. The following should minimize confusion caused by the differences.

• X and Y coordinates are the same in both systems.

• Values of the trigonometric functions remain unchanged:

a.	Quadrant I:	sin + cos +
b.	Quadrant II:	sin + cos −
c.	Quadrant III:	sin − cos −
d.	Quadrant IV:	sin − cos +

• If the direction is alpha (a) in the surveyor's system and theta (Θ) in the math/science system, they are related by:

$a = 90° - \Theta$ and $\Theta = 90° - a$

$\sin a = \cos\Theta$ and $\cos a = \sin\Theta$.

The polar/rectangular (P/R) conversion in most calculators is hardwired to give:

$D \cos\Theta$ = change in X (departure) and

$D \sin\Theta$ = change in Y (latitude).

The same result (departure and latitude) is obtained in the surveyor's system by using:

$D \sin a$ = change in X (departure) and

$D \cos a$ = change in Y (latitude).

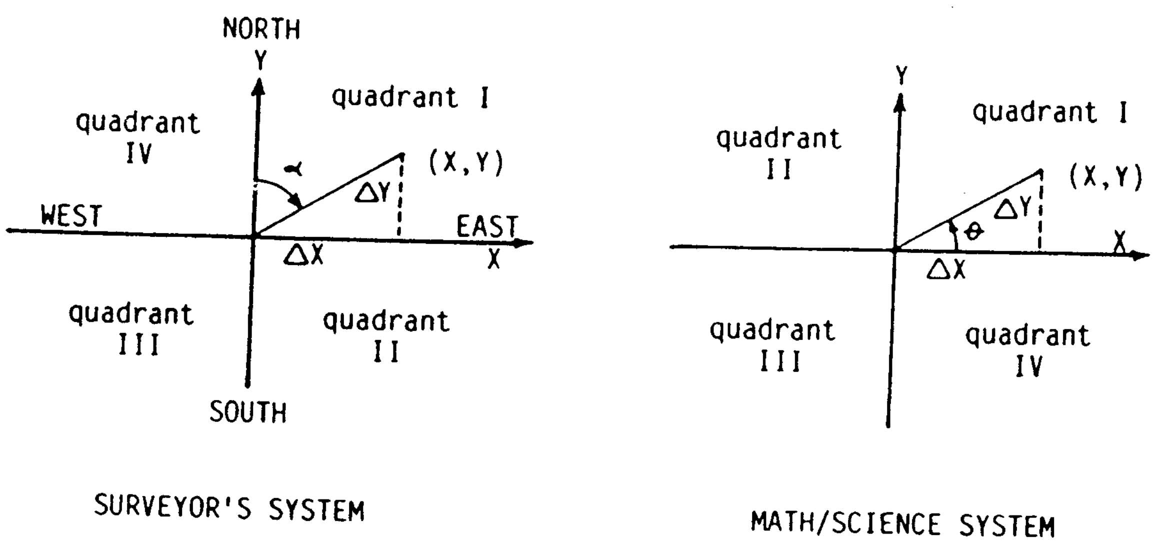

Figure 1. Comparison of coordinate systems.

Since the calculator does not know the difference between a and Θ, the only change required of the user is to switch the latitude and departure designators associated with polar/rectangular conversion. For example, to go from polar to rectangular coordinates, the calculator manual may say departure is displayed as the product of distance times cosine of direction entered. If the direction were entered as an azimuth in the surveyor's system, the same product is really the course latitude rather than the departure. A similar switch is made going from rectangular to polar. If one inputs the departure/latitude where the manual asks for latitude/departure (math/science system) the resulting azimuth will be correct in the surveyor's reference system.

The coordinate computation elements used throughout this paper and shown in Figure 2 are:

X_1 & Y_1 = X and Y coordinates of beginning point occupied.

X_2 & Y_2 = X and Y coordinates of ending point.

X_p & Y_p = X and Y coordinates of intermediate point defined by the intersection of:

a. two lines (line-line).

b. a line with a circle (line-circle).

c. two circles (circle-circle).

a_0 = Direction (azimuth) *from* point 1 *to* point 2.

a = Generic direction *from* point 1 *to* any point.

β = Direction *from* intersection point *to* point 2.

D_0 = Distance *from* point 1 *to* point 2.

D_1 = Distance *from* point 1 *to* intersection point.

D_2 = Distance *from* intersection point *to* point 2.

$\Delta X = X_2 - X_1$ (departure of course 1 to 2).

$\Delta Y = Y_2 - Y_1$ (latitude of course 1 to 2).

γ = Angle formed at point 1 by D_0 and D_1 (always +).

Assumptions and Approach

The following assumptions and philosophy are critical to understanding derivation and use of equations listed in the Summary of Coordinate Computation Formulas later in this paper.

• Coordinates of a point are considered primary data. If coordinates for a point are not available, the direction and distance to it from some known point are the defining data for that point. However, once established, the coordinates are primary data and all other quantities are derived from the coordinates.

• Uncertainty, random errors, positional tolerance and standard deviation are not considered. This paper deals only with consistency of geometrical elements of a problem and redundancy is used only to check correctness of a solution.

• Inasmuch as state plane coordinates have large magnitudes it is desirable to use coor-

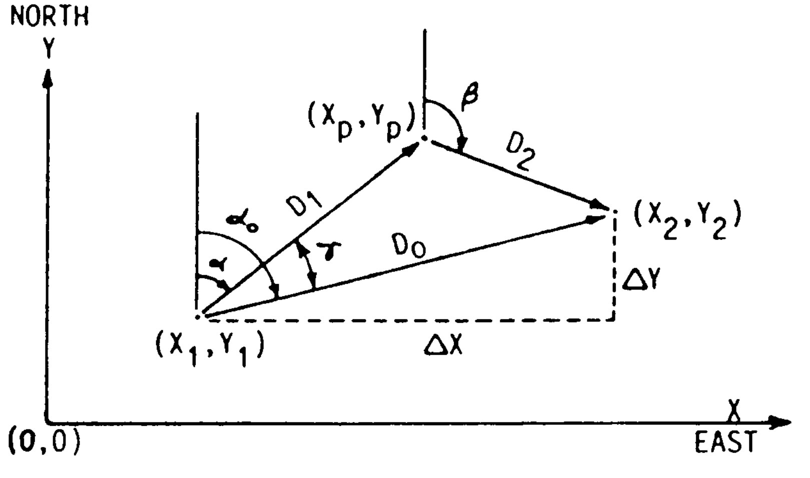

Figure 2. Elements of coordinate computation.

dinate differences. Certain problems with significant figures and calculator capacity are avoided if a trigonometric function is multiplied by a coordinate difference rather than a very large number.

- The approach for the intersection solutions is to write the forward computation symbolically, once for each course. The resulting equations are solved for an unknown direction or distance as required to compute coordinates of the intersection point from point 1 using the forward computation. An inverse from there to point 2 will give a direction and distance which can be compared with given data on the same course. If the check fails, an error was made and the computation must be repeated.

- More steps than might be necessary are included in an effort to make the derivation easy to follow.

Basic Formulas

Forward computation formulas are very basic, but are the basis of intersection formula derivation. Referring to Figure 2 and following conventions previously adopted:

$$X_2 = X_1 + D_o \sin a_o = X_1 + \Delta X \quad (1)$$

$$Y_2 = Y_1 + D_o \cos a_o = Y_1 + \Delta Y \quad (2)$$

When one uses the P/R (polar-rectangular) key on a calculator, it computes ΔX and ΔY using direction and distance provided by the user. Note however, if the calculator is hardwired to the math/science system, it gives

$\Delta X = D_o$ cos (direction) and $\Delta Y = D_o$ sin (direction).

If data were input in the surveyor's system (azimuth from north), the desired computation is still performed but result is given as:

$\Delta Y = D_o$ cos (azimuth) and $\Delta X = D_o$ sin (azimuth).

Thus, if one switches latitude/departure designations, the P/R key can be very useful. When programming, use of a summation key makes the P/R even more powerful if the programmer and/or user is willing to keep track of which registers are accumulated as latitudes and which are accumulated as departures.

The inverse computations are also basic formulas which are hardwired into most calculators. Given coordinates of two points, equations (1) and (2) are used as:

$$\Delta X = D_o \sin a_o = X_2 - X_1 \quad (3)$$

$$\Delta Y = D_o \cos a_o = Y_2 - Y_1 \quad (4)$$

The inverse computation uses equations (3) and (4) to find direction and distance between two points. The distance is obtained by squaring and adding equations (3) and (4):

$$\Delta X^2 + \Delta Y^2 = D_o^2(\sin^2 a_o + \cos^2 a_o)$$

$$= (X_2 - X_1)^2 + (Y_2 - Y_1)^2$$

$$D_o = \sqrt{\Delta X^2 + \Delta Y^2}$$

$$= \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} \quad (5)$$

Dividing equation (3) by (4) will give azimuth point 1 to point 2:

$$(\Delta X/\Delta Y) = (D_o \sin a_o / D_o \sin a_o) = \tan a_o \quad (6)$$

The relationship given in equation (6) is always true, but will not yield a unique azimuth (0° to 360°) due to the repetitive nature of the tangent function. Another problem in a longhand solution is that computing an azimuth of due east or west is undefined when ΔY is zero. These problems are handled in the longhand solution by adding a very small value (0.000001) to ΔY before dividing and by using bearings for direction. However, a unique azimuth can be found efficiently if one is willing to use the following tests.

- If ΔY is negative, then
$$a_o = 180° + \arctan(\Delta X/\Delta Y)$$
- If test 1 fails *and* if ΔX is negative, then
$$a_o = 360° + \arctan(\Delta X/\Delta Y)$$
- If test 1 and test 2 *both* fail, then
$$a_o = \arctan(\Delta X/\Delta Y).$$

It is rarely necessary to use the preceding test as most calculators have the R/P (rectangular-polar) conversion built-in. Given ΔX and ΔY the R/P key will provide a distance and a unique direction even if ΔY is zero. If an azimuth in the surveyor's system is desired, one must be careful to input ΔX where the calculator expects latitude and ΔY for the departure. Otherwise, if hardwired in the math/science system, the calculator will give a counterclockwise azimuth from east. If a negative azimuth is encountered, one can execute the "mod" function found on some

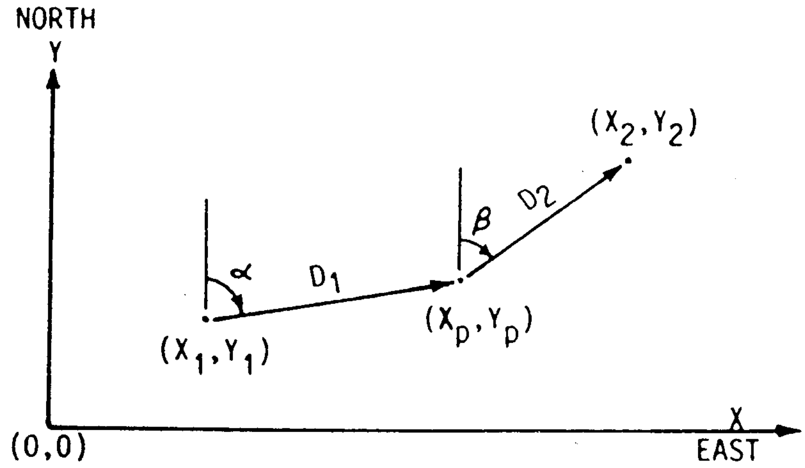

Figure 3. General case for intersection computation.

calculators or simply add 360° to put the azimuth in the proper range.

Intersections

So far, only two points have been considered. Intersections involve three points, the beginning and ending point plus the intermediate intersection point. Figure 3 illustrates the general intersection case written with the forward computation formulas as:

$$X_p = X_1 + D_1 \sin a \quad (7)$$
$$Y_p = Y_1 + D_1 \cos a \quad (8)$$
$$X_2 = X_p + D_2 \sin\beta \quad (9)$$
$$Y_2 = Y_p + D_2 \cos\beta \quad (10)$$

This system of four equations can be solved for any combination of four unknowns. For intersections, point 1 and point 2 are always known and the coordinates of the intersection are always unknown. Different intersection problems are defined by various combinations of unknowns as shown in Table 1. Unknowns (X_p, Y_p) are eliminated from the set of four equations by solving (9) for X_p and equating to (7) and solving (10) for Y_p and equating to (8).

$$X_p = X_1 + D_1 \sin a = X_2 - D_2 \sin\beta \quad (11)$$
$$Y_p = Y_1 + D_1 \cos a = Y_2 - D_2 \cos\beta \quad (12)$$

Utilizing coordinate differences the equations are written as:

$$\Delta X = X_2 - X_1 = D_1 \sin a + D_2 \sin\beta \quad (13)$$
$$\Delta Y = Y_2 - Y_1 = D_1 \cos a + D_2 \cos\beta \quad (14)$$

The problem is now reduced to two equations which can be solved to find that pair of unknowns required by the particular intersection.

Line-Line Intersection

Given: (X_1, Y_1), (X_2, Y_2), a and β.
Find: (X_p, Y_p), D_1 and D_2.

The approach is to solve (14) for D_2, substitute into (13) and solve for D_1. Knowing D_1 and a coordinates (X_p, Y_p) are computed using forward position formula given by equations (7) and (8). Knowing coordinates of the intersection point, distance D_2 can be computed using the inverse computation. The inverse direction from the intersection point to point 2 should agree identically (within significant digit capacity of calculator) with β, the given azimuth for course 2. If the inverse di-

Table 1. Different intersection problems defined by various combinations of unknowns.

Known	*Always Unknown*	*Unique Unknown*	*Intersection*
a & β	X_p & Y_p	D_1 & D_2	line-line
a & D_2	X_p & Y_p	D_1 & β	line-circle
D_1 & D_2	X_p & Y_p	a & β	circle-circle

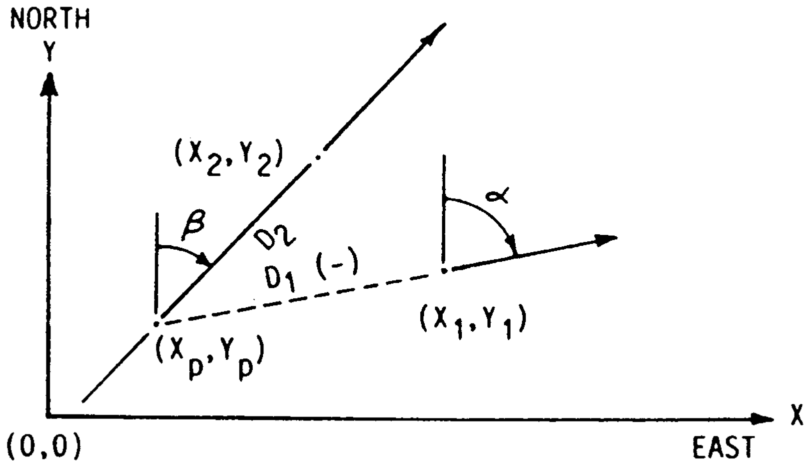

Figure 4. Example of negative distance in line-line intersection.

rection does not agree, an error was made in the computations.

From equation (14),

$$D_2 = (\Delta Y - D_1 \cos a)/\cos\beta$$

Substituting into equation (13) and solving for D_1,

$$\Delta X = D_1 \sin a + [(\Delta Y - D_1 \cos a)/\cos\beta] \sin\beta$$

$$D_1 \sin a = (\Delta X \cos\beta - \Delta Y \sin\beta + D_1 \cos a \sin\beta) /\cos\beta$$

$$D_1 (\sin a\cos\beta - \cos a\sin\beta) = \Delta X \cos\beta - \Delta Y \sin\beta$$

$$D_1 = (\Delta X \cos\beta - \Delta Y \sin\beta)/\sin(a - \beta) \qquad (15)$$

Equation (15) is an expression for D_1 in terms of coordinate differences between points 1 and 2 and the directions (azimuths) of the two lines. The only restriction on the solution is that the two lines not be parallel. If they are parallel, they will never intersect and no solution can be found for D_1 due to dividing by zero. Note that D_1 may be either positive or negative. If D_1 is negative, as shown in Figure 4, it means the intersection occurs behind you in the sense of forward being in the direction a. In summary:

$$\Delta X = X_2 - X_1 \ \&\ \Delta Y = Y_2 - Y_1 \qquad (3) \text{ and } (4)$$

$$D_1 = (\Delta X \cos\beta - \Delta Y \sin\beta)/\sin(a - \beta) \qquad (15)$$

$$X_p = X_1 + D_1 \sin a \qquad (7)$$

$$Y_p = Y_1 + D_1 \cos a \qquad (8)$$

$$D_2 = \sqrt{(X_2 - X_p)^2 + (Y_2 - Y_p)^2} \qquad (5)$$

$\tan\beta = (X_2 - X_p)/(Y_2 - Y_p)$ to check given value β. (6)

Line-Circle Intersection

Given: (X_1,Y_1), (X_2,Y_2), a and D_2.

Find: (X_p,Y_p), D_1 and β.

The approach in this case is to solve equation (14) for $\cos\beta$ and to use a form of it in equation (13) to solve for D_1. As shown in Figure 5, two values of D_1 are expected. Thus it

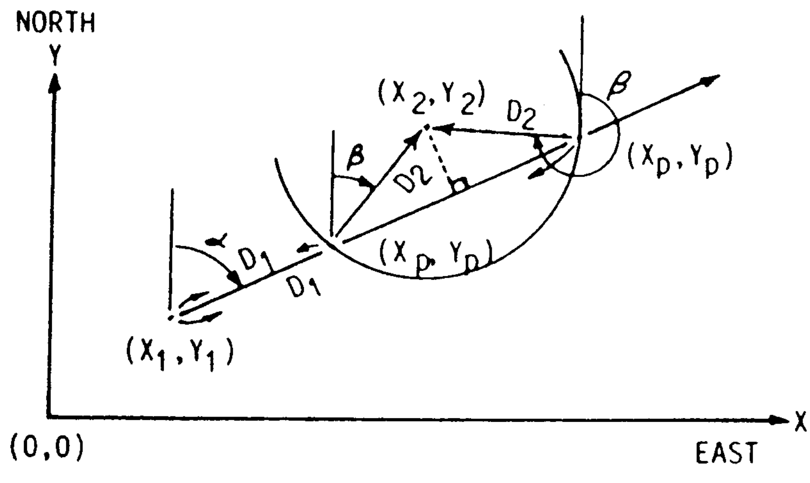

Figure 5. Elements of line-circle intersection.

is no surprise that the solution involves a quadratic equation. From equation (14),

$$\cos\beta = (\Delta Y - D_1 \cos a)/D_2$$

$$\cos^2\beta = (\Delta Y^2 - 2\Delta Y D_1 \cos a + D_1^2 \cos^2 a)/D_2^2 \quad (16)$$

Recall the trigonometric identity:

$$\sin\beta = \sqrt{1 - \cos^2\beta} \quad (17)$$

Now substitute equation (16) into (17), then into (13) to get:

$$\Delta X = D_1 \sin a + D_2 \sqrt{1 - (\Delta Y^2 - 2\Delta Y D_1 \cos a + D_1^2 \cos^2 a)/D_2^2}$$

and

$$(\Delta X - D_1 \sin a)^2 = D_2^2[1 - (\Delta Y^2 - 2\Delta Y D_1 \cos a + D_1^2 \cos^2 a)/D_2^2]$$

from which

$$\Delta X^2 - 2\Delta X D_1 \sin a + D_1^2 \sin^2 a - D_2^2 + \Delta Y^2 - 2\Delta Y D_1 \cos a + D_1^2 \cos^2 a = 0.$$

Collect D_1^2 and D_1 terms in quadratic form,

$$a(D_1^2) + b(D_1) + c = 0$$

$$D_1^2(\sin^2 a + \cos^2 a) + D_1(-2\Delta X \sin a - 2\Delta Y \cos a) + \Delta X^2 + \Delta Y^2 - D_2^2 = 0 \quad (18)$$

Equating coefficients in equation (18) with those of the quadratic,

$a = 1$, $b = -2\Delta X \sin a - 2\Delta Y \cos a$ and

$c = \Delta X^2 + \Delta Y^2 - D_2^2$.

Substituting values into the quadratic equation solution gives:

$$D_1 = \Delta X \sin a + \Delta Y \cos a \pm \sqrt{(b^2/4) - c} \quad (19)$$

The terms under the radical are:

$$(b^2/4) - c = (4\Delta X^2 \sin^2 a + 8\Delta X \sin a \Delta Y \cos a + 4\Delta Y^2 \cos^2 a)/4 - \Delta X^2 - \Delta Y^2 + D_2^2$$
$$= \Delta X^2(\sin^2 a - 1) + \Delta Y^2(\cos^2 a - 1) + 2\Delta X \sin a \Delta Y \cos a + D_2^2$$

Now recall that

$\sin^2\Theta - 1 = -\cos^2\Theta$ and $\cos^2\Theta - 1 = \sin^2\Theta$.

Therefore,

$$(b^2/4) - c = -1(\Delta X^2 \cos^2 a - 2\Delta X \cos a \Delta Y \sin a + \Delta Y^2 \sin^2 a) + D_2^2$$
$$= D_2^2 - (\Delta X \cos a - \Delta Y \sin a)^2 \quad (20)$$

Combining equations (19) and (20) we get:

$$D_1 = \Delta X \sin a + \Delta Y \cos a \pm \sqrt{D_2^2 - (\Delta X \cos a - \Delta Y \sin a)^2} \quad (21)$$

Equation (21) is an expression for D_1 in terms of coordinate differences from point 1 to point 2 and the direction of the line (a) from point 1 to the intersection point. Note that two values of D_1 were obtained as expected.

In addition to efficiency enjoyed by using equation (21), there is an unexpected bonus obtained from an analysis of the value under the radical. If the value under the radical is negative, the line does not intersect the circle and no intersection can be computed. If the

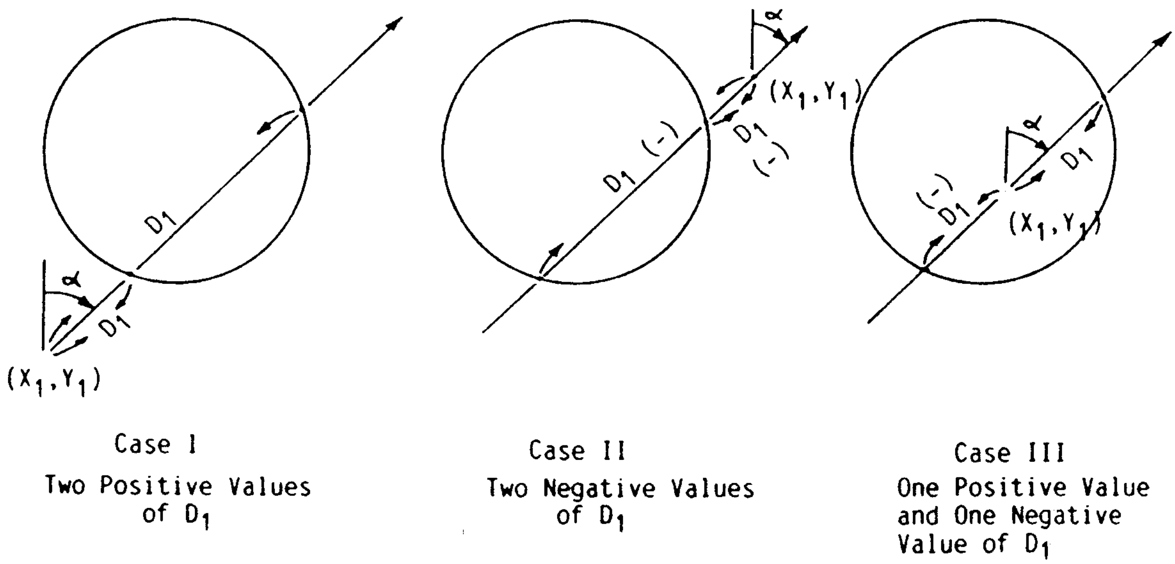

Figure 6. Example of positive and negative values of D_1 in line-circle intersection.

value under the radical is exactly 0, there is only one solution—the tangent case. All positive values under the radical yield two possible values of D_1, one for each intersection. Note in Figure 6 that values of D_1 which are negative are just as legitimate as positive values. The formula for D_1 is very powerful in that it tells if there is no solution, one solution, or two solutions. Additionally, the values of D_1 can be examined to see if one or both of the intersections are "behind" us. There are no restrictions other than making sure the line intersects the circle and being careful if the exact tangent solution is desired. In that case the perpendicular offset formula will give a specific value of D_2 given two points and the direction of the line. In summary:

$$\Delta X = X_2 - X_1 \; \& \; \Delta Y = Y_2 - Y_1 \qquad (3) \text{ and } (4)$$

$$D_1 = \Delta X \sin a + \Delta Y \cos a \pm \sqrt{D_2^2 - (\Delta X \cos a - \Delta Y \sin a)^2} \qquad (21)$$

$$X_p = X_1 + D_1 \sin a \qquad (7)$$

$$Y_p = Y_1 + D_1 \cos a \qquad (8)$$

$$\tan\beta = (X_2 - X_p)/(Y_2 - Y_p) \qquad (6)$$

$$D_2 = \sqrt{(X_2 - X_p)^2 + (Y_2 - Y_p)^2}, \text{ used to check computation.} \qquad (5)$$

Circle-Circle Intersection

Given: (X_1,Y_1), (X_2,Y_2), D_1 & D_2

Find: (X_p,Y_p), a & β

The simple elegance of the line-line and line-circle intersection solutions justified the laborious algebra required to obtain them. That is not so with the circle-circle intersection. If we were to start with equations (13) and (14), eliminate β as we did to get equation (18) and try to solve it for a, the derivation gets very messy. However, our goal can be met by using the "longhand" approach and programming the result. The solution is direct, simple, rigorous, and efficient.

Refer to Figure 7 showing two intersecting circles; one with its center at point 1, the other at point 2. The approach will be to inverse from point 1 to point 2 to obtain the direction (a_o) and distance (D_o). The angle (γ) at point 1 is computed using three side lengths in the law of cosines. The azimuth to the intersection points is obtained by adding angle (γ) to or subtracting it from the inverse direction (a_o). Coordinates of each intersection are then computed using the forward computation. An inverse from the intersection point to point 2 will give the direction (β) and distance (D_2) which can be used as a check. In summary:

$$a_o = \text{inverse direction from point 1 to point 2.} \qquad (5)$$

$$D_o = \text{inverse distance from point 1 to point 2.} \qquad (6)$$

$$\cos\gamma = (D_1^2 + D_o^2 - D_2^2)/(2D_1D_o) \qquad (22)$$

$$a = a_o \pm \gamma \text{ (two solutions)} \qquad (23)$$

$$X_p = X_1 + D_1 \sin a \qquad (7)$$

$$Y_p = Y_1 + D_1 \cos a \qquad (8)$$

$$\tan\beta = (X_2 - X_p)/(Y_2 - Y_p) \qquad (6)$$

$$D_2 = \sqrt{(X_2 - X_p)^2 + (Y_2 - Y_p)^2}, \text{ used to check computation.} \qquad (5)$$

Consider possible alternatives. If the

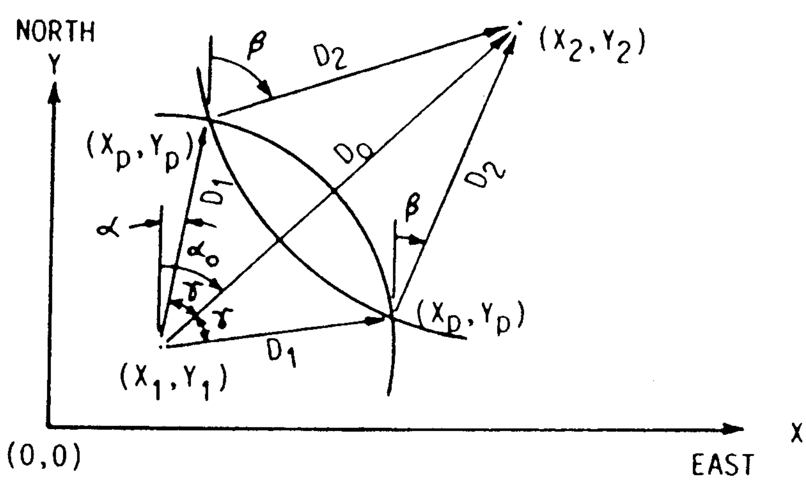

Figure 7. Elements of circle-circle intersection.

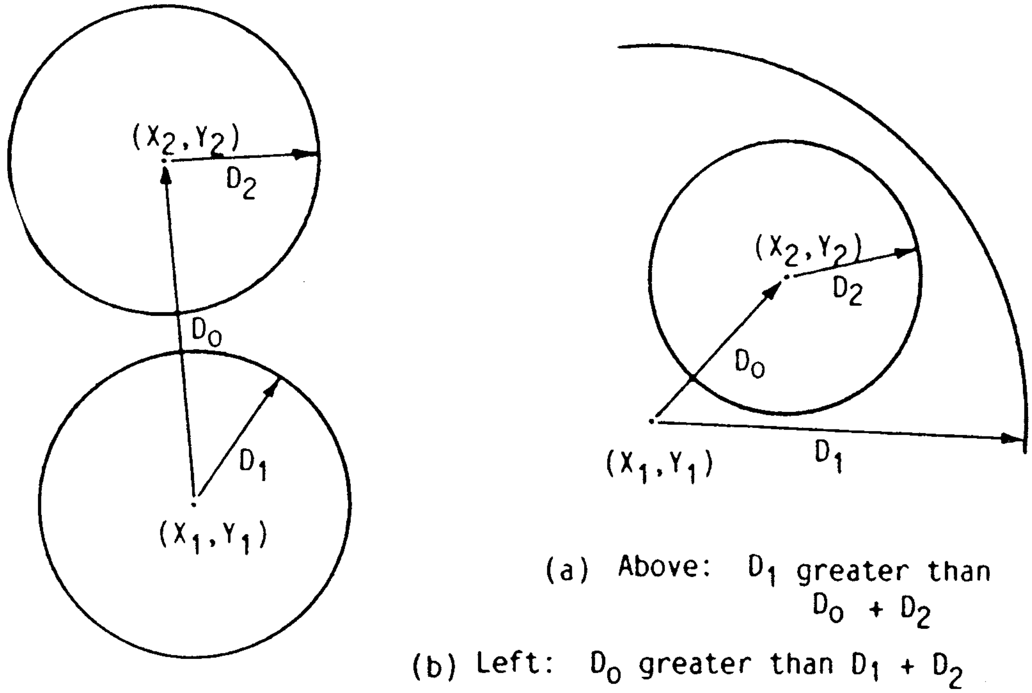

Figure 8. Failure of circles to intersect.

two circles do not intersect as shown in Figure 8a or 8b, $|\cos \gamma|$ will be greater than 1.0 for which (γ) does not exist. If the two circles are tangent at exactly one point, $\cos\gamma$ will be either -1.0 or 1.0 and (γ) will be exactly 0° or 180°. Otherwise, two solutions exist. The angle (γ) is added to or subtracted from the inverse direction (a_o) to give an azimuth to be used with D_1 and the forward computation formula. The inverse is then used to give (β) and to check the given distance D_2.

Perpendicular Offset

Given: (X_1,Y_1), (X_2,Y_2) and a

Find: D_2

In some cases it is desired to know only the perpendicular distance from a line to a given point. A line-line intersection with $\beta = a + 90°$ will give the complete solution, but if coordinates of the intersection are not needed and distance D_2 (as shown in Figure 9) is the only item of interest, a simple equation can be used.

The approach is to solve equations (13) and (14) for D_2 using $\beta = a + 90°$. Recall trigonometric identities.

$$\sin(\Theta + 90°) = \cos\Theta \quad \& \quad \cos(\Theta + 90°) = -\sin\Theta$$

Therefore, equations (13) and (14) can be written as:

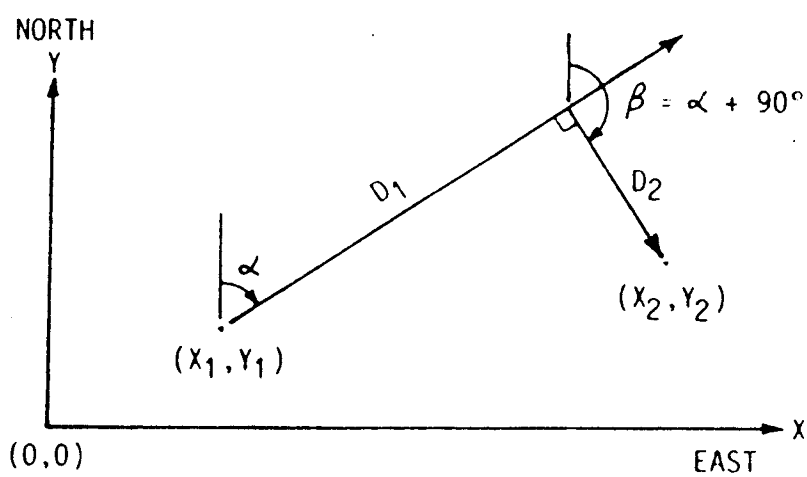

Figure 9. Elements of perpendicular offset.

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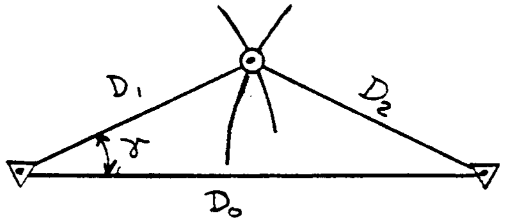
FAILURE OF CIRCLES TO INTERSECT

I. SEPARATED FROM ONE ANOTHER:



$$\cos\gamma = \frac{D_1^2 + D_o^2 - D_2^2}{2 D_1 D_o}$$

IF $D_1 + D_2$ IS LESS THAN D_o THE CIRCLES WILL NOT INTERSECT.

THE LIMIT OCCURS WHEN $D_1 + D_2 = D_o$ EXACTLY OR

$$D_2 = D_o - D_1 \quad \&$$

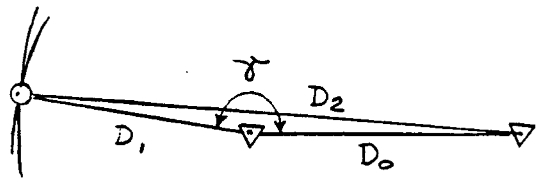
$$D_2^2 = D_o^2 - 2 D_o D_1 + D_1^2$$

FOR TANGENT (LIMITING) CASE

$$\cos\gamma = \frac{D_1^2 + D_o^2 - (D_o^2 - 2 D_o D_1 + D_1^2)}{2 D_1 D_o}$$

$$= \frac{2 D_1 D_o}{2 D_1 D_o} = 1.0000$$

II. WHEN ONE CIRCLE IS ENTIRELY WITHIN THE OTHER:



IF D_2 IS GREATER THAN $D_1 + D_o$ THE CIRCLES WILL NOT INTERSECT.

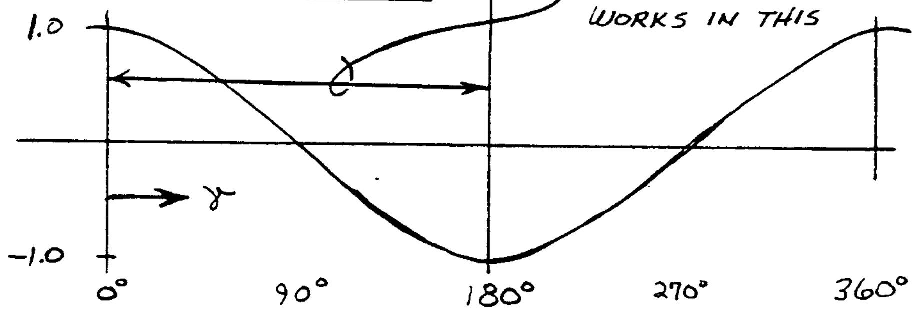
THE LIMIT OCCURS WHEN $D_1 + D_o = D_2$ EXACTLY, OR

$$D_2^2 = D_1^2 + 2 D_1 D_o + D_o^2$$

FOR TANGENT (LIMITING) CASE:

$$\cos\gamma = \frac{D_1^2 + D_o^2 - (D_1^2 + 2 D_1 D_o + D_o^2)}{2 D_1 D_o} = \frac{-2 D_1 D_o}{2 D_1 D_o} = -1.0000$$

GRAPH OF COSINE CURVE



THE ARC COSINE FUNCTION OF A CALCULATOR WORKS IN THIS RANGE 0°–180°.

THE ANGLE γ WILL ALWAYS BE POSITIVE AND LESS THAN OR EQUAL TO 180°

$$\Delta X = D_1 \sin a + D_2 \sin(a + 90°) = D_1 \sin a + D_2 \cos a \quad (24)$$

$$\Delta Y = D_1 \cos a + D_2 \cos(a + 90°) = D_1 \cos a - D_2 \sin a \quad (25)$$

Solve equation (25) for D_1 and substitute into equation (24) to solve for D_2.

$$D_1 = (\Delta Y + D_2 \sin a)/\cos a$$

$$\Delta X = [(\Delta Y + D_2 \sin a)/\cos a] + D_2 \cos a$$

$$\Delta X \cos a = \Delta Y \sin a + D_2(\sin^2 a + \cos^2 a)$$

$$D_2 = \Delta X \cos a - \Delta Y \sin a \quad (26)$$

Equation (26) for a perpendicular offset distance is elegant, rigorous, and simple. Not only does it give the offset distance, but which side of the line it is on is given by whether it comes out positive or negative. If point 2 lies right of the line as assumed in the derivation, D_2 comes out positive. However, if the point lies left of the line, D_2 comes out negative. This feature can be particularly useful when computing offset from a random traverse line to a section line for clearing and marking.

One final item about the perpendicular offset. Note that the perpendicular offset distance given by equation (26) also appears under the radical of equation (21) as one of the legs of a right triangle within the circle. Thus, equation (26) might be programmed as a subroutine to be called as required.

Since programmable calculators have become available the author has encountered several inadequate intersection programs which are or have been available on a commercial basis. In one specific case, the program failed entirely because the programmer did not consider restrictions imposed by his assumptions. In other cases, the accuracy of the solution suffers because a large state plane coordinate value is multiplied by a trigonometric function rather than a coordinate difference as presented herein.

Who is responsible for integrity of surveying computations? Is it the technician pushing the buttons as directed by the boss? Is it the person who signs off on the computations or plat? Is it the person who writes and/or markets the programs? Or is it those who teach? Assuming all share that responsibility, it is hoped this systematic approach to coordinate computation and use of programmable calculators will improve our collective professional efforts.

Summary of Coordinate Computation Formulas

Forward:

(1) $X_2 = X_1 + D_o \sin a_o$

(2) $Y_2 = Y_1 + D_o \cos a_o$

Inverse: (Figure 10)

(3) $\Delta X = X_2 - X_1$

(4) $\Delta Y = Y_2 - Y_1$

(5) $D_o = \sqrt{\Delta X^2 + \Delta Y^2}$

(6) $\tan a_o = (\Delta X / \Delta Y)$

$a_o = \arctan(\Delta X/\Delta Y)$, Quadrant I

$a_o = 180° + \arctan(\Delta X/\Delta Y)$, Quadrants II & III

$a_o = 360° + \arctan(\Delta X/\Delta Y)$, Quadrant IV

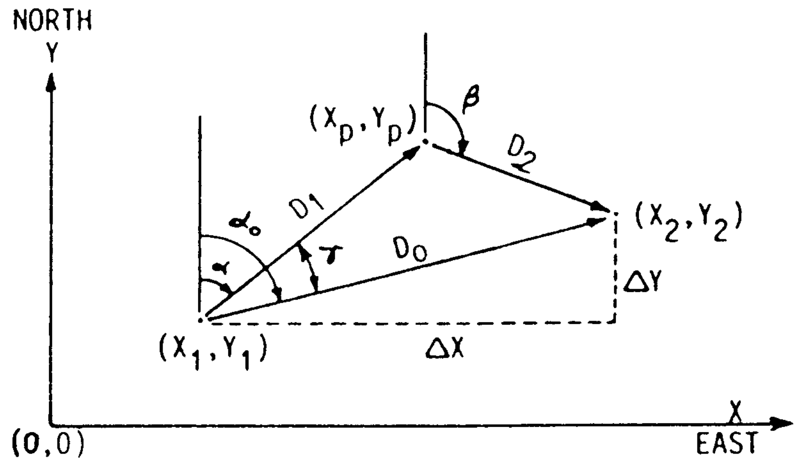


Figure 10. Elements of coordinate computation.

Intersections:

(15) Line-Line

$$D_1 = (\Delta X \cos\beta - \Delta Y \sin\beta)/\sin(a-\beta)$$

(21) Line-Circle

$$D_1 = \Delta X \sin a + \Delta Y \cos a \pm \sqrt{D_2^2 - (\Delta X \cos a - \Delta Y \sin a)^2}$$

(22) Circle-Circle

$$\cos\gamma = (D_1{}^2 + D_o{}^2 - D_2{}^2)/(2D_1D_o)$$

(23) $a = a_o \pm \gamma$

(7) $X_p = X_1 + D_1 \sin a$

(8) $Y_p = Y_1 + D_1 \cos a$

β = azimuth from inverse (check line-line intersection)

D_2 = inverse distance (check line-circle & circle-circle)

Perpendicular Offset:

(26) $D_2 = \Delta X \cos a - \Delta Y \sin a$ ■