Magical Least Squares - or
When is One Least Squares Adjustment Better Than Another?

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Introduction

Least squares adjustment, that is, the sum of the squares of the residuals will be a minimum, has been proven and accepted as the best possible method for adjusting survey data.

Within reason, it is also true that, depending upon how weights are selected, you can get any answer you want using least squares. Therefore, the issue in discussing "How good are my results?" switches from the choice of the tool (least squares) to how the tool is used. Of course, the input data must first be checked and verified blunder-free.

Given blunder-free survey data and a specific statement of how weights are selected, all least squares packages should provide the same answers. Differences from one brand software to another will have to do with the survey data input (formats, weights etc) and what information is included in the report after the adjustment is completed. This article looks at the differences caused by using 3 different weighting assumptions on a small network.

The example used in this paper is a GPS network based upon two A-order HARN points. Station "Reilly" is located in the central horseshoe of the NMSU campus and Station "Crucesair" is located at the Las Cruces airport some 16 kilometers west of campus. The network consists of 7 independent baselines connecting 4 additional points to the existing HARN stations as shown in Figure 1.

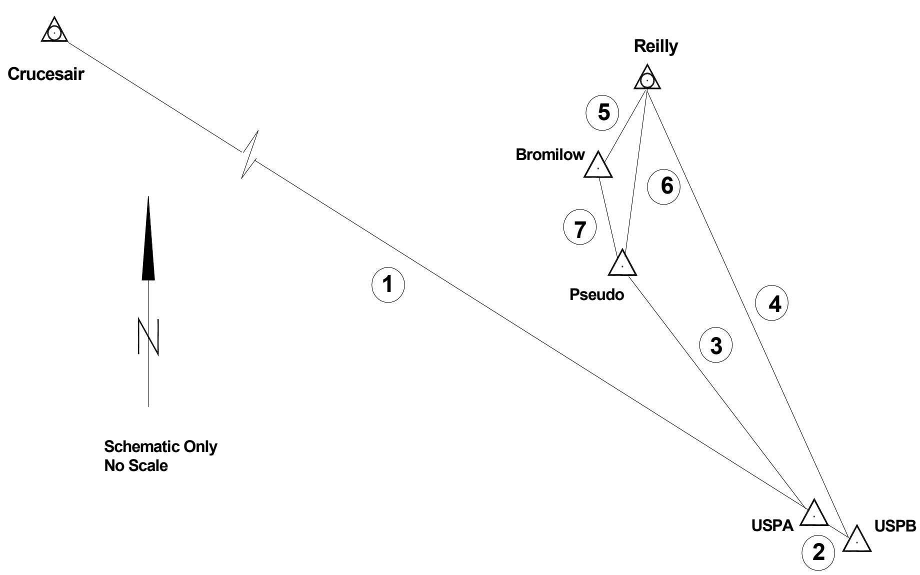

Figure 1 GPS Network at NMSU

The GPS baselines shown and used were collected on four different dates over a period of 5 years. These are not the only baselines on campus nor are they the only observations between the points in question. These baselines were selected because they show excellent consistency, are independent, and include often used points. The results are also used to show the difference between network accuracy and local accuracy.

Control Values and Observed Vectors

The NAD83 geocentric X/Y/Z coordinates for A-order HARN stations "Reilly" and "Crucesair" are as published by the National Geodetic Survey (NGS) and were held fixed in this exercise. They are:

Station Reilly	Station Crucesair
X = -1,556,177.615 m	X = -1,571,430.672 m
Y = -5,169,235.319 m	Y = -5,164,782.312 m
Z = 3,387,551.709 m	Z = 3,387,603.188 m

Single frequency Trimble GPS receivers were used to collect static data, 57 minutes being the shortest common observation time for any of the 7 baselines. The baseline components and the covariance matrix for each observed baseline as determined by Trimble software using default processing parameters are:

Baseline 1 – Crucesair to USPA – observed 3/28/02 (use subscript CA):

			Sxx	Syy	Szz
ΔX_{CA} =	15,752.080 m	Sxx	6.321492E-06		
ΔY_{CA} =	-5,179.102 m	Syy	1.545948E-05	4.739877E-05	
ΔZ_{CA} =	-903.089 m	Szz	-1.061303E-05	-3.184780E-05	2.388036E-05

Baseline 2 – USPA to USPB – observed 11/12/03 (use subscript AB):

			Sxx	Syy	Szz
ΔX_{AB} =	14.964 m	Sxx	1.412453E-06		
ΔY_{AB} =	-15.365 m	Syy	1.285418E-06	4.653209E-06	
ΔZ_{AB} =	-16.664 m	Szz	-5.669127E-07	-1.658118E-06	1.872469E-06

Baseline 3 – USPA to Pseudo – observed 3/28/02 (use subscript AP):

			Sxx	Syy	Szz
ΔX_{AP} =	-528.036 m	Sxx	9.505016E-08		
ΔY_{AP} =	560.657 m	Syy	8.957064E-08	3.729339E-07	
ΔZ_{AP} =	585.897 m	Szz	-5.022282E-08	-2.221975E-07	3.363763E-07

Baseline 4 – USPB to Reilly - observed 3/28/02 (use subscript BR):

			Sxx	Syy	Szz
ΔX_{BR} =	-514.003 m	Sxx	3.650165E-07		
ΔY_{BR} =	741.438 m	Syy	9.024127E-07	2.796189E-06	
ΔZ_{BR} =	868.293 m	Szz	-6.189027E-07	-1.881145E-06	1.410196E-06

Baseline 5 – Bromilow to Reilly - observed 12/10/98 (use subscript MR):

			Sxx	Syy	Szz
ΔX_{MR} =	32.134 m	Sxx	2.762550E-07		
ΔY_{MR} =	51.175 m	Syy	3.200312E-07	6.870545E-07	
ΔZ_{MR} =	94.198 m	Szz	-2.008940E-07	-4.006259E-07	4.661596E-07

Baseline 6 – Pseudo to Reilly – observed 1/23/02 (use subscript PR):

				Sxx	Syy	Szz
ΔX_{PR} =	29.000 m	Sxx		1.325760E-07		
ΔY_{PR} =	165.422 m	Syy		1.317165E-07	5.265054E-07	
ΔZ_{PR} =	265.719 m	Szz		-7.253348E-08	-3.020965E-07	5.006575E-07

Baseline 7 – Bromilow to Pseudo – observed 1/23/02 (use subscript MP):

				Sxx	Syy	Szz
ΔX_{MP} =	3.136 m	Sxx		3.367818E-07		
ΔY_{MP} =	-114.242 m	Syy		3.937476E-07	8.766570E-07	
ΔZ_{MP} =	-171.527 m	Szz		-5.186521E-07	-8.977932E-07	1.446501E-06

Blunder Checks

In order to verify the absence of blunders in the baselines, misclosures are computed for each component (X/Y/Z) as follows:

Traverse including baselines 1, 2, and 4 (from “Crucesair” to “Reilly”):

	X	**Y**	**Z**
Station Crucesair	-1,571,430.672 m	-5,164,782.312 m	3,387,603.188 m
Baseline 1	15,752.080 m	-5,179.102 m	-903.089 m
Baseline 2	14.964 m	-15.365 m	-16.664 m
Baseline 4	-514.003 m	741.438 m	868.293 m
Computed value	-1,556,177.631 m	-5,169,235.341 m	3,387,551.728 m
Station Reilly	-1,556,177.615 m	-5,169,235.319 m	3,387,551.709 m
Misclosures	**-0.016 m**	**-0.022 m**	**0.019 m**

Loop including baselines 2-3-7-5-4 (being careful to preserve sign convention):

Baseline 2	-14.964 m	15.365 m	16.664 m
Baseline 3	-528.036 m	560.657 m	585.897 m
Baseline 7	-3.136 m	114.242 m	171.527 m
Baseline 5	32.134 m	51.175 m	94.198 m
Baseline 4	514.003 m	-741.438 m	-868.293 m
Misclosures	**0.001 m**	**0.001 m**	**-0.007 m**

Loop including baselines 5-6-7 (being careful to preserve sign convention):

Baseline 5	32.134 m	51.175 m	94.198 m
Baseline 6	-29.000 m	-165.422 m	-265.719 m
Baseline 7	-3.136 m	114.242 m	171.527 m
Misclosures	**-0.002 m**	**-0.005 m**	**0.006 m**

All baselines have been included in the checks and all misclosures are acceptable. Therefore, it is legitimate to perform a least squares adjustment of the 7 baselines to determine the “best” adjusted position for points USPA, USPB, Pseudo, and Bromilow. Any adjustment should also provide information on the quality of the answers, i.e., “What is the standard deviation of the computed position?” - in both the geocentric (X/Y/Z) reference frame and in the local (east/north/up) reference frame. This paper uses 3 different weighting schemes and shows a comparison of the various answers.

Adjustment Model and Procedure

Although there may be variations, the following GPS network solution formulation has been accepted and is often used. Matrices are shown in **bold type**.

1. Each baseline includes 3 observations – one each in ΔX, ΔY, and ΔZ. In this case, there are 7 baselines, so the solution will include 21 observation equations.

2. Each new point has an unknown position; X_i, Y_i, and Z_i. There will be 12 answers (parameters) for the 4 new points. It takes 1 observation to find 1 parameter. The "extra" observations are known (in statistics) as degrees-of-freedom. In surveying terms, extra observations are known as redundancies. In this case, r = 21 – 12 = 9.

3. Equations for each of the 7 baselines (21 equations in all) are written in the form:

$$X_{there} = X_{here} + \Delta X_{here\ to\ there} + \text{a residual}$$
$$Y_{there} = Y_{here} + \Delta Y_{here\ to\ there} + \text{a residual}$$
$$Z_{there} = Z_{here} + \Delta Z_{here\ to\ there} + \text{a residual}$$

   a. A residual, v_i, is given whatever value it takes to make each equation "correct."
   b. The GPS observations are ΔX_i, ΔY_i, and ΔZ_i for each of the 7 baselines.
   c. The coordinate value of X, Y, and Z of "here" or "there" may be known or unknown. If known, the value is combined with the GPS observed ΔX_i, ΔY_i, or ΔZ_i as in the following solution section.
   d. If the value of X, Y, or Z at a named station is unknown, it is called a parameter. A symbol and subscript, Δ_i, is used to represent each parameter.

4. A least squares matrix solution requires the equations to be written as $\mathbf{v} + \mathbf{B}\Delta = \mathbf{f}$. Once the problem is written in this form, the matrix solution is obtained as:

   $\Delta = (\mathbf{B}^t \mathbf{W} \mathbf{B})^{-1} \mathbf{B}^t \mathbf{W} \mathbf{f}$ or stated differently, $\Delta = \mathbf{N}^{-1} \mathbf{B}^t \mathbf{W} \mathbf{f}$. where;

   Δ is a vector of parameters (answers) - not to be confused with $\Delta X/\Delta Y/\Delta Z$.
   N is a matrix of normal equations. $\mathbf{N}^{-1}$ contains statistics for the answers.
   W is the weight matrix. Various values of **W** is the issue in this paper.
   B is a matrix of coefficients for the unknown parameters.
   f is a vector of constants computed from known values and observations
   v is a vector of residuals as noted above.

5. Once the parameters are known, the residuals are computed as:

$$\mathbf{v} = \mathbf{f} - \mathbf{B}\Delta$$

6. The estimated (a posteriori) reference variance is computed as:

   $(\text{sigma}_0\ \text{hat})^2 = (\mathbf{v}^t \mathbf{W} \mathbf{v}) / r$, where r is the redundancy.

7. The covariance matrix of the computed positions in the X/Y/Z reference frame, is computed as:

$$\mathbf{\Sigma}_{X/Y/Z} = (\text{sigma}_0\ \text{hat})^2\ \mathbf{N}^{-1}$$

8. The standard deviations of the computed X, Y, Z coordinates are the square roots of the diagonal elements of the $\mathbf{\Sigma_{X/Y/Z}}$ covariance matrix. The local reference frame standard deviations at each point are computed as the square root of the diagonal elements after the geocentric covariance matrix has been rotated to the local e/n/u (right-handed) reference frame using the following rotation matrix.

$$\mathbf{\Sigma_{e/n/u}} = \mathbf{R}\,\mathbf{\Sigma_{X/Y/Z}}\,\mathbf{R}^t \quad \text{where,} \quad \mathbf{R} = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix}$$

With regard to the rotation matrix above, latitude is positive north of the equator and negative in the southern hemisphere. Longitude is counted as positive 0° to 360° eastward from the Greenwich Meridian. West longitude (the convention used in North America) is used as a negative number.

9. The latitude and longitude for the point must be computed from the geocentric X/Y/Z values of the point. The longitude computation is straightforward but the latitude requires iteration. The longitude (0° to 360° eastward) is computed as:

$$\tan\lambda = \frac{Y}{X} \quad \text{or, with due regard to signs and quadrants,} \quad \lambda = \tan^{-1}\left(\frac{Y}{X}\right)$$

The equation for latitude is:

$$\tan\phi = \frac{Z}{P}\left(1 + \frac{e^2 N \sin\phi}{Z}\right) \qquad \text{where} \quad P = \sqrt{X^2 + Y^2},$$

but the problem is that the unknown variable, φ, is on both sides of the equation. That means the equation must be solved by iteration. The first approximation is computed as:

$$\phi_0 = \arctan\left(\frac{Z}{P(1-e^2)}\right) \quad \text{and} \quad N_0 = \frac{a}{\sqrt{1 - e^2 \sin^2\phi_0}}.$$

These values of φ_0 and N_0 are then used to compute another value of φ and N as:

$$\phi_1 = \arctan\left[\frac{Z}{P}\left(1 + \frac{e^2 N_0 \sin\phi_0}{Z}\right)\right] \quad \text{and} \quad N_1 = \frac{a}{\sqrt{1 - e^2 \sin^2\phi_1}}$$

A second latitude, φ_2, is computed using the equation above but with updated φ and N. The iteration continues until the value of latitude changes less than some tolerance selected by the user. Typically, no more than 2 or 3 iterations are required to obtain a latitude accurate enough (say within 0.001 seconds) to be used in the rotation matrix. Other methods for performing the iteration can also be used.

The Solution

Following the procedure outlined above, the first step is to write the observation equations in standard form, then re-write them in the specific form of $\mathbf{v + B\Delta = f}$. The subscripts suggested earlier are used. The constant **f** vector is computed from known values on the right side of the equals sign in the matrix solution formulation.

	Observation form	Matrix Solution form	Constant = f
1.	$X_A = X_C + \Delta X_{CA} + v_1$	$v_1 - X_A = -X_C - \Delta X_{CA}$	$= f_1 =$ 1,555,678.592
2.	$Y_A = Y_C + \Delta Y_{CA} + v_2$	$v_2 - Y_A = -Y_C - \Delta Y_{CA}$	$= f_2 =$ 5,169,961.414
3.	$Z_A = Z_C + \Delta Z_{CA} + v_3$	$v_3 - Z_A = -Z_C - \Delta Z_{CA}$	$= f_3 =$ -3,386,700.099
4.	$X_B = X_A + \Delta X_{AB} + v_4$	$v_4 + X_A - X_B = -\Delta X_{AB}$	$= f_4 =$ -14.964
5.	$Y_B = Y_A + \Delta Y_{AB} + v_5$	$v_5 + Y_A - Y_B = -\Delta Y_{AB}$	$= f_5 =$ 15.365
6.	$Z_B = Z_A + \Delta Z_{AB} + v_6$	$v_6 + Z_A - Z_B = -\Delta Z_{AB}$	$= f_6 =$ 16.664
7.	$X_P = X_A + \Delta X_{AP} + v_7$	$v_7 + X_A - X_P = -\Delta X_{AP}$	$= f_7 =$ 528.036
8.	$Y_P = Y_A + \Delta Y_{AP} + v_8$	$v_8 + Y_A - Y_P = -\Delta Y_{AP}$	$= f_8 =$ -560.657
9.	$Z_P = Z_A + \Delta Z_{AP} + v_9$	$v_9 + Z_A - Z_P = -\Delta Z_{AP}$	$= f_9 =$ -585.897
10.	$X_R = X_B + \Delta X_{BR} + v_{10}$	$v_{10} + X_B = X_R - \Delta X_{BR}$	$= f_{10} =$ -1,555,663.612
11.	$Y_R = Y_B + \Delta Y_{BR} + v_{11}$	$v_{11} + Y_B = Y_R - \Delta Y_{BR}$	$= f_{11} =$ -5,169,976.757
12.	$Z_R = Z_B + \Delta Z_{BR} + v_{12}$	$v_{12} + Z_B = Z_R - \Delta Z_{BR}$	$= f_{12} =$ 3,386,683.416
13.	$X_R = X_M + \Delta X_{MR} + v_{13}$	$v_{13} + X_M = X_R - \Delta X_{MR}$	$= f_{13} =$ -1,556,209.749
14.	$Y_R = Y_M + \Delta Y_{MR} + v_{14}$	$v_{14} + Y_M = Y_R - \Delta Y_{MR}$	$= f_{14} =$ -5,169,286.494
15.	$Z_R = Z_M + \Delta Z_{MR} + v_{15}$	$v_{15} + Z_M = Z_R - \Delta Z_{MR}$	$= f_{15} =$ 3,387,457.511
16.	$X_R = X_P + \Delta X_{PR} + v_{16}$	$v_{16} + X_P = X_R - \Delta X_{PR}$	$= f_{16} =$ -1,556,206.615
17.	$Y_R = Y_P + \Delta Y_{PR} + v_{17}$	$v_{17} + Y_P = Y_R - \Delta Y_{PR}$	$= f_{17} =$ -5,169,400.741
18.	$Z_R = Z_P + \Delta Z_{PR} + v_{18}$	$v_{18} + Z_P = Z_R - \Delta Z_{PR}$	$= f_{18} =$ -3,387,285.990
19.	$X_P = X_M + \Delta X_{MP} + v_{19}$	$v_{19} + X_M - X_P = -\Delta X_{MP}$	$= f_{19} =$ -3.136
20.	$Y_P = Y_M + \Delta Y_{MP} + v_{20}$	$v_{20} + Y_M - Y_P = -\Delta Y_{MP}$	$= f_{20} =$ 114.242
21.	$Z_P = Z_M + \Delta Z_{MP} + v_{21}$	$v_{21} + Z_M - Z_P = -\Delta Z_{MP}$	$= f_{21} =$ 171.527

The **B** matrix is shown below and the elements are obtained as the partial derivatives of the unknown parameters in the equations above. In this case, the partial derivative values are either 0 or 1. The **f** matrix is a vector of constants as computed above. The weight matrix, **W**, is the matrix to be changed for the different solutions. The matrix manipulations to find the solution are based upon those three matrices **B**, **W** and **f**.

For the first solution, the weight matrix is taken to be identity (1's on the diagonal and 0's otherwise). That means all observations are of equal weight. The second solution uses the standard deviation of the observed ΔX, ΔY, ΔZ of each baseline. Each weight is $1/\sigma^2$ of the observation and appears on the diagonal of the weight matrix. The off-diagonal elements are all 0. The third option uses the full covariance matrix of each baseline to determine the weights. Using the full covariance matrix utilizes the correlation between baseline components and should provide a better network solution.

Using the equations above, the problem is written in the **v + BΔ = f** matrix format. Note, the values in the B matrix are all 0's and 1's (partial derivatives). The constant f is a vector computed in the equations above. The symbol, f_i , is used below rather than the actual values due to margin space limitations.

V **+** **B** **Δ** **=** **f**

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \\ v_{16} \\ v_{17} \\ v_{18} \\ v_{19} \\ v_{20} \\ v_{21} \end{bmatrix}
+
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} X_A \\ Y_A \\ Z_A \\ X_B \\ Y_B \\ Z_B \\ X_P \\ Y_P \\ Z_P \\ X_M \\ Y_M \\ Z_M \end{bmatrix}
=
\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \\ f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{16} \\ f_{17} \\ f_{18} \\ f_{19} \\ f_{20} \\ f_{21} \end{bmatrix}
$$

The first solution will utilize the identity matrix as the weight matrix. It has 1's on the diagonal and 0's everywhere else. The 21x21 weight matrix for equal weights is shown on the next page.

As stated previously, the solution to find the answers (unknown parameters) is the matrix manipulation:

$\Delta = (\mathbf{B^t\,W\,B})^{-1}\,\mathbf{B^t\,W\,f}$ Note, $\mathbf{N}^{-1} = (\mathbf{B^t\,W\,B})^{-1}$. It is used later.

Option 1 of 3 – Equal Weights

$$W = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}$$

The matrix operations can be performed by various math software packages including an excel spreadsheet, MathCAD, and others. The "Matrix" program which comes with the textbook, Elementary Surveying (10^{th} or 11^{th} Ed) was used in this example. "Matrix" is also available free as a download from the Penn State Surveying program web site - http://surveying.wb.psu.edu.

The answers for the parameters (X/Y/Z coordinates for the unknown points) are:

	USPA	USPB	Pseudo	Bromilow
X	-1,555,678.5843 m	-1,555,663.6161 m	-1,556,206.6167 m	-1,556,209.7508 m
Y	-5,169,961.4037 m	-5,169,976.7628 m	-5,169,400.7423 m	-5,169,286.4971 m
Z	3,386,700.0922 m	3,386,683.4221 m	3,387,285.9885 m	3,387,457.5132 m

Using these values for the unknown parameters, the computation continues to find each residual and the estimated reference variance. Using "Matrix" computational tools, the residuals are computed as:

$$\mathbf{v} = \mathbf{f} - \mathbf{B}\,\Delta$$

The values of the residuals and the length of the associated baselines are:

	Residual	Baseline	Length
ΔX	0.0077 m		
ΔY	0.0103 m	Crucesair to USPA	16,606 m
ΔZ	-0.0068 m		

ΔX	0.0041 m		
ΔY	0.0059 m	USPA to USPB	27 m
ΔZ	-0.0061 m		
ΔX	0.0036 m		
ΔY	0.0044 m	USPA to Pseudo	968 m
ΔZ	-0.0007 m		
ΔX	0.0041 m		
ΔY	0.0059 m	USPB to Reilly	1,252 m
ΔZ	-0.0061 m		
ΔX	0.0019 m		
ΔY	0.0031 m	Bromilow to Reilly	112 m
ΔZ	-0.0022 m		
ΔX	0.0017 m		
ΔY	0.0013 m	Pseudo to Reilly	314 m
ΔZ	0.0015 m		
ΔX	-0.0019 m		
ΔY	-0.0031 m	Bromilow to Pseudo	206 m
ΔZ	0.0022 m		

Using these residuals, the estimated (a posteriori) reference variance is:

$$(\text{sigma}_0\ \text{hat})^2 = \mathbf{v^t\, W\, v} \;/\; r = 0.00046538 / 9 = 0.00005171$$

The standard deviation of each X/Y/Z coordinate is obtained from $(\text{sigma}_0\ \text{hat})^2\ \mathbf{N^{-1}}$ where the variance of each parameter is on the diagonal of the $\mathbf{N^{-1}}$ matrix.

USPA:
$\sigma_X = \sqrt{(0.00005171 * 0.47619)} = 0.0050$ m
$\sigma_Y = \sqrt{(0.00005171 * 0.47619)} = 0.0050$ m
$\sigma_Z = \sqrt{(0.00005171 * 0.47619)} = 0.0050$ m

USPB:
$\sigma_X = \sqrt{(0.00005171 * 0.61905)} = 0.0056$ m
$\sigma_Y = \sqrt{(0.00005171 * 0.61905)} = 0.0056$ m
$\sigma_Z = \sqrt{(0.00005171 * 0.61905)} = 0.0056$ m

Pseudo:
$\sigma_X = \sqrt{(0.00005171 * 0.47619)} = 0.0050$ m
$\sigma_Y = \sqrt{(0.00005171 * 0.47619)} = 0.0050$ m
$\sigma_Z = \sqrt{(0.00005171 * 0.47619)} = 0.0050$ m

Bromilow:
$\sigma_X = \sqrt{(0.00005171 * 0.61905)} = 0.0056$ m
$\sigma_Y = \sqrt{(0.00005171 * 0.61905)} = 0.0056$ m
$\sigma_Z = \sqrt{(0.00005171 * 0.61905)} = 0.0056$ m

The $\mathbf{N^{-1}}$ matrix below is a 12 x 12 symmetrical matrix that contains the cofactors of the parameters. When multiplied by the estimated reference variance, $(\text{sigma}_0\ \text{hat})^2$, each diagonal element becomes the variance of the parameter. The standard deviation of each parameter is the square root of its variance. See computations above.

$$
\begin{bmatrix}
0.47619 & 0 & 0 & 0.23810 & 0 & 0 & 0.19048 & 0 & 0 & 0.09524 & 0 & 0 \\
0 & 0.47619 & 0 & 0 & 0.23810 & 0 & 0 & 0.19048 & 0 & 0 & 0.09524 & 0 \\
0 & 0 & 0.47619 & 0 & 0 & 0.23810 & 0 & 0 & 0.19048 & 0 & 0 & 0.09524 \\
0.23810 & 0 & 0 & 0.61905 & 0 & 0 & 0.09524 & 0 & 0 & 0.04762 & 0 & 0 \\
0 & 0.23810 & 0 & 0 & 0.61905 & 0 & 0 & 0.09524 & 0 & 0 & 0.04762 & 0 \\
0 & 0 & 0.23810 & 0 & 0 & 0.61905 & 0 & 0 & 0.09524 & 0 & 0 & 0.04762 \\
0.19048 & 0 & 0 & 0.09524 & 0 & 0 & 0.47619 & 0 & 0 & 0.23810 & 0 & 0 \\
0 & 0.19048 & 0 & 0 & 0.09524 & 0 & 0 & 0.47619 & 0 & 0 & 0.23810 & 0 \\
0 & 0 & 0.19048 & 0 & 0 & 0.09524 & 0 & 0 & 0.47619 & 0 & 0 & 0.23810 \\
0.09524 & 0 & 0 & 0.04762 & 0 & 0 & 0.23810 & 0 & 0 & 0.61905 & 0 & 0 \\
0 & 0.09524 & 0 & 0 & 0.04762 & 0 & 0 & 0.23810 & 0 & 0 & 0.61905 & 0 \\
0 & 0 & 0.09524 & 0 & 0 & 0.04762 & 0 & 0 & 0.23810 & 0 & 0 & 0.61905
\end{bmatrix}
$$

Next, the computed X/Y/Z position of each new point is used to compute the latitude and longitude of the point. Then, those geodetic positions are used to rotate the X/Y/Z reference frame standard deviations to the local reference frame. Note, the first 3 x 3 sub matrix in $\mathbf{N}^{-1}$ is associated with point USPA, the second 3 x 3 sub matrix is associated with USPB, the third 3 x 3 sub matrix is associated with Pseudo, and the last 3 x 3 sub matrix is associated with Bromilow. In each case the 3 x 3 sub matrix is a diagonal matrix with 0's on the off-diagonal. Also note other off-diagonal 3 x 3 sub matrices reflect correlation between points. That covariance information is used to compute local accuracies between points and is described in a subsequent section.

The alogrithm for rotating standard deviations from one reference frame to another is given in steps 8 and 9 above. The BURKORD9 software (gratis from the author) was used to compute the latitude and longitude (and ellipsoid height) of each point and to rotate the geocentric standard deviations to the local reference frame. The results are:

	Geocentric Coordinates and sigma		Geodetic Coordinates and local sigma	
Station USPA:				
X =	-1,555,678.584 m	+/- 0.0050 m	φ = 32° 16' 23."00012 N	+/- 0.0050 m (N)
Y =	-5,169,961.404 m	+/- 0.0050 m	λ = 106° 44' 48."90828 W	+/- 0.0050 m (E)
Z =	3,386,700.092 m	+/- 0.0050 m	h = 1,178.025 m	+/- 0.0050 m (U)
Station USPB:				
X =	-1,555,663.616 m	+/- 0.0056 m	φ = 32° 16' 22."36248 N	+/- 0.0056 m (N)
Y =	-5,169,976.763 m	+/- 0.0056 m	λ = 106° 44' 48."19160 W	+/- 0.0056 m (E)
Z =	3,386,683.422 m	+/- 0.0056 m	h = 1,177.912 m	+/- 0.0056 m (U)

Station Pseudo:

X =	-1,556,206.617 m +/- 0.0050 m	φ = 32º 16' 45."74650 N	+/- 0.0050 m (N)
Y =	-5,169,400.742 m +/- 0.0050 m	λ = 106º 45' 14."39978 W	+/- 0.0050 m (E)
Z =	3,387,285.988 m +/- 0.0050 m	h = 1,165.644 m	+/- 0.0050 m (U)

Station Bromilow:

X =	-1,556,209.751 m +/- 0.0056 m	φ = 32º 16' 52."33408 N	+/- 0.0056 m (N)
Y =	-5,169,286.497 m +/- 0.0056 m	λ = 106º 45' 15."77275 W	+/- 0.0056 m (E)
Z =	3,386,457.513 m +/- 0.0056 m	h = 1,165.525 m	+/- 0.0056 m (U)

Note that, when using equal weights, the uncertainty is spherical. That is, the standard deviations at a point are the same for each component and remain the same when rotated to the local reference frame. That will not be true for options 2 and 3.

Option Two – Weights by Standard Deviation of Baseline Components

For the option 2, the weight matrix is based upon the variance (standard deviation squared) of each component of each baseline. The matrix is 21 rows x 21 columns but the values are only on the diagonal. All off-diagonal elements are 0. Therefore, only the diagonal elements are listed and determined individually as σ_0^2 / σ^2. The user may choose any value for σ_0^2. If σ_0^2 is chosen = 1.0 as below, then $w_{i,i}$ = 1 / variance.

Baseline	Component	Variance	Element	Weight
	ΔX:	6.321492E-06	$w_{1,1}$	158,190.5
Crucesair to USPA	ΔY:	4.739877E-05	$w_{2,2}$	21,097.6
	ΔZ:	2.388036E-05	$w_{3,3}$	41,875.4
	ΔX:	1.412453E-06	$w_{4,4}$	707,988.2
USPA to USPB	ΔY:	4.653209E-06	$w_{5,5}$	214,905.4
	ΔZ:	1.872469E-06	$w_{6,6}$	534,054.2
	ΔX:	9.505016E-08	$w_{7,7}$	10,520,760.8
USPA to Pseudo	ΔY:	3.729339E-07	$w_{8,8}$	2,681,440.3
	ΔZ:	3.363763E-07	$w_{9,9}$	2,972,861.0
	ΔX:	3.650165E-07	$w_{10,10}$	2,739,602.2
USPB to Reilly	ΔY:	2.796189E-06	$w_{11,11}$	357,629.6
	ΔZ:	1.410196E-06	$w_{12,12}$	709,121.3
	ΔX:	2.762550E-07	$w_{13,13}$	3,619,844.0
Bromilow to Reilly	ΔY:	6.870545E-07	$w_{14,14}$	1,455,488.6
	ΔZ:	4.661596E-07	$w_{15,15}$	2,145,188.0
	ΔX:	1.325760E-07	$w_{16,16}$	7,542,843.4
Pseudo to Reilly	ΔY:	5.265054E-07	$w_{17,17}$	1,899,315.7
	ΔZ:	5.006575E-07	$w_{18,18}$	1,997,373.4
	ΔX:	3.367818E-07	$w_{19,19}$	2,969,281.6
Bromilow to Pseudo	ΔY:	8.766570E-07	$w_{20,20}$	1,140,696.7
	ΔZ:	1.446501E-06	$w_{21,21}$	691,323.4

An incidental comment is that weights are relative. The user may choose any value for σ_0^2. That means that each weight could be multiplied by the same number such that the smallest weight is 1.00. Similarly, all the weights could be multiplied by another number such that the largest weight is 1.00. In either case, it would not change the answer or the statistics, but it would change the magnitude of the numbers that appear in the $\mathbf{N}^{-1}$ matrix. Using the very large weights as shown above means that the numbers in the $\mathbf{N}^{-1}$ matrix below are quite small. That is why the $\mathbf{N}^{-1}$ values are written in scientific notation.

Using the same **B** matrix and **f** vector as before along with the revised weight matrix, the answers for the parameters (option 2 X/Y/Z coordinates for the unknown points) are:

	USPA	USPB	Pseudo	Bromilow
X	-1,555,678.579 m	-1,555,663.612 m	-1,556,206.615 m	-1,556,209.750 m
Y	-5,169,961.396 m	-5,169,976.759 m	-5,169,400.740 m	-5,169,286.496 m
Z	3,386,700.090 m	3,386,683.420 m	3,387,285.988 m	3,387,457.512 m

Computing the residuals as before but using the updated values of the parameters, the values of the residuals and the length of the associated baselines are:

Residual		Baseline	Length
ΔX:	0.0132 m		
ΔY:	0.0174 m	Crucesair to USPA	16,606 m
ΔZ:	-0.0092 m		
ΔX:	0.0022 m		
ΔY:	0.0028 m	USPA to USPB	27 m
ΔZ:	-0.0056 m		
ΔX:	0.0001 m		
ΔY:	-0.0001 m	USPA to Pseudo	968 m
ΔZ:	0.0009 m		
ΔX:	0.0006 m		
ΔY:	0.0017 m	USPB to Reilly	1,252 m
ΔZ:	-0.0042 m		
ΔX:	0.0008 m		
ΔY:	0.0016 m	Bromilow to Reilly	112 m
ΔZ:	-0.0009 m		
ΔX:	-0.0003 m		
ΔY:	-0.0014 m	Pseudo to Reilly	314 m
ΔZ:	0.0023 m		
ΔX:	-0.0009 m		
ΔY:	-0.0020 m	Bromilow to Pseudo	206 m
ΔZ:	0.0028 m		

Using these residuals, the estimated (a posteriori) reference variance is:

$$(\text{sigma}_0\ \text{hat})^2 = \mathbf{v}^t\,\mathbf{W}\,\mathbf{v} \;/\; r = 111.587 / 9 = 12.3986$$

The standard deviation of each X/Y/Z coordinate is obtained from $(\text{sigma}_0\ \text{hat})^2 \times \mathbf{N}^{-1}$ where the variance of each parameter is computed from the diagonal element of $\mathbf{N}^{-1}$.

$$\left[\begin{array}{cccccccccccc}
1.779E-07 & 0 & 0 & 3.653E-08 & 0 & 0 & 9.503E-08 & 0 & 0 & 4.282E-08 & 0 & 0 \\
0 & 6.852E-07 & 0 & 0 & 2.572E-07 & 0 & 0 & 3.520E-07 & 0 & 0 & 1.546E-07 & 0 \\
0 & 0 & 5.846E-7 & 0 & 0 & 2.512E-07 & 0 & 0 & 3.164E-07 & 0 & 0 & 7.712E-08 \\
3.653E-08 & 0 & 0 & 2.976E-07 & 0 & 0 & 1.951E-08 & 0 & 0 & 8.794E-09 & 0 & 0 \\
0 & 2.572E-07 & 0 & 0 & 1.843E-06 & 0 & 0 & 1.321E-07 & 0 & 0 & 5.805E-08 & 0 \\
0 & 0 & 2.512E-07 & 0 & 0 & 9.123E-07 & 0 & 0 & 1.359E-07 & 0 & 0 & 3.313E-08 \\
9.503E-08 & 0 & 0 & 1.951E-08 & 0 & 0 & 1.015E-07 & 0 & 0 & 4.576E-08 & 0 & 0 \\
0 & 3.520E-07 & 0 & 0 & 1.321E-07 & 0 & 0 & 3.723E-07 & 0 & 0 & 1.636E-07 & 0 \\
0 & 0 & 3.164E-07 & 0 & 0 & 1.359E-07 & 0 & 0 & 3.533E-07 & 0 & 0 & 8.610E-08 \\
4.282E-08 & 0 & 0 & 8.794E-09 & 0 & 0 & 4.576E-08 & 0 & 0 & 1.724E-07 & 0 & 0 \\
0 & 1.546E-07 & 0 & 0 & 5.805E-08 & 0 & 0 & 1.636E-07 & 0 & 0 & 4.571E-07 & 0 \\
0 & 0 & 7.712E-08 & 0 & 0 & 3.313E-08 & 0 & 0 & 8.610E-08 & 0 & 0 & 3.735E-07
\end{array}\right]$$

USPA:
$\sigma_X = \sqrt{(12.3986 * 1.779E-07)} = 0.0015$ m
$\sigma_Y = \sqrt{(12.3986 * 6.852E-07)} = 0.0029$ m
$\sigma_Z = \sqrt{(12.3986 * 5.846E-07)} = 0.0027$ m

USPB:
$\sigma_X = \sqrt{(12.3986 * 2.976E-07)} = 0.0019$ m
$\sigma_Y = \sqrt{(12.3986 * 1.843E-06)} = 0.0048$ m
$\sigma_Z = \sqrt{(12.3986 * 9.123E-07)} = 0.0034$ m

Pseudo:
$\sigma_X = \sqrt{(12.3986 * 1.015E-07)} = 0.0011$ m
$\sigma_Y = \sqrt{(12.3986 * 3.723E-07)} = 0.0021$ m
$\sigma_Z = \sqrt{(12.3986 * 3.533E-07)} = 0.0021$ m

Bromilow:
$\sigma_X = \sqrt{(12.3986 * 1.724E-07)} = 0.0015$ m
$\sigma_Y = \sqrt{(12.3986 * 4.571E-07)} = 0.0024$ m
$\sigma_Z = \sqrt{(12.3986 * 3.735E-07)} = 0.0022$ m

As before, the X/Y/Z position of each new point is used to compute the latitude and longitude of the point. Then, those geodetic positions are used to rotate the X/Y/Z reference frame standard deviations to the local reference frame. The results are:

Geocentric Coordinates and sigma	Geodetic Coordinates and local sigma

Station USPA:

X = -1,555,678.579 m +/- 0.0015 m	φ = 32° 16' 23."00020 N +/- 0.0027 m (N)
Y = -5,169,961.396 m +/- 0.0029 m	λ = 106° 44' 48."90816 W +/- 0.0017 m (E)
Z = 3,386,700.090 m +/- 0.0027 m	h = 1,178.016 m +/- 0.0028 m (U)

Station USPB:

X =	-1,555,663.612 m +/- 0.0019 m	φ = 32° 16' 22."36251 N	+/- 0.0038 m (N)
Y =	-5,169,976.759 m +/- 0.0048 m	λ = 106° 44' 48."19151 W	+/- 0.0023 m (E)
Z =	3,386,683.420 m +/- 0.0034 m	h = 1,177.907 m	+/- 0.0041 m (U)

Station Pseudo:

X =	-1,556,206.615 m +/- 0.0011 m	φ = 32° 16' 45."74653 N	+/- 0.0021 m (N)
Y =	-5,169,400.740 m +/- 0.0021 m	λ = 106° 45' 14."39974 W	+/- 0.0012 m (E)
Z =	3,387,285.988 m +/- 0.0021 m	h = 1,165.641 m	+/- 0.0020 m (U)

Station Bromilow:

X =	-1,556,209.750 m +/- 0.0015 m	φ = 32° 16' 52."33407 N	+/- 0.0022 m (N)
Y =	-5,169,286.496 m +/- 0.0024 m	λ = 106° 45' 15."77273 W	+/- 0.0016 m (E)
Z =	3,386,457.512 m +/- 0.0022 m	h = 1,165.523 m	+/- 0.0023 m (U)

Note that the adjusted X/Y/Z coordinate values changed slightly but that the standard deviations – both in the X/Y/Z reference frame and the local components – are quite a bit smaller. Clearly, using the standard deviations for weighting the observations has improved the statistics of the adjusted positions.

Also note that the standard deviations are different for each component – both in the X/Y/Z reference frame and in the local reference frame. Error ellipses are derived from the local northing and easting standard deviations.

Option Three – Weights by the Full Baseline Covariance Matrix

For option 3, the weight matrix is computed from the covariance matrix of each observed baseline. As before, the weight matrix is a 21 x 21 matrix and the weights are computed as the inverse of the covariance matrices of the baselines.

A bit more explanation is in order. In option 1, the weights were all identical – 1's on the diagonal of the weight matrix. In option 2, we chose the weights to be 1/ variance of each baseline component (again just on the diagonal – all off-diagonal elements were 0's). In option 3, the weights are defined in the same manner, but the details are not quite so simple because we need to use the covariance information for each baseline. In all three options, the definition of the weight matrix is:

$$\mathbf{W} = \sigma_0^2 \, \mathbf{\Sigma}^{-1}$$

In all three cases, we chose σ_0^2 = 1.00. If we had picked another value for σ_0^2, we could have "scaled" the weights so that the largest weight was 1.00 or so that the smallest weight was 1.00. Remember, weights are relative but the magnitude of the numbers is determined by the user's choice of σ_0^2.

In option 3, the weight matrix (21 x 21) is the inverse of the (21 x 21) covariance matrix and the covariance matrix is built using 7 individual baseline (3 x 3) covariance matrices as shown on the next page. Note, it is not permissible to just take the reciprocal of each variance element as we did in option 2 (because it was a diagonal matrix), but one must compute the inverse of the entire **Σ** matrix as shown below. Yes, because these were treated as independent baselines, the weight matrix will be a diagonal matrix of seven 3 x 3 sub matrices and the remaining off-diagonal elements are 0's. If the baselines are not independent, the off-diagonal elements will contain non-zero values to show correlation between baselines. Such correlation occurs when data collected at one station is used in processing two (or more) baselines. That case is not considered here.

Using the same **B** matrix and **f** vector as before along with the revised weight matrix (as shown on subsequent pages, the third set of answers for the parameters (X/Y/Z coordinates for the unknown points) is:

	USPA	USPB	Pseudo	Bromilow
X	-1,555,678.579 m	-1,555,663.613 m	-1,556,206.615 m	-1,556,209.750 m
Y	-5,169,961.396 m	-5,169,976.761 m	-5,169,400.740 m	-5,169,286.496 m
Z	3,386,700.089 m	3,386,683.419 m	3,387,285.987 m	3,387,457.512 m

Computing the residuals as before but using the updated values of the parameters, the values of the residuals and the length of the associated baselines are:

	Residual	Baseline	Length
ΔX:	0.0128 m		
ΔY:	0.0178 m	Crucesair to USPA	16,606 m
ΔZ:	-0.0101 m		
ΔX:	0.0018 m		
ΔY:	0.0002 m	USPA to USPB	27 m
ΔZ:	-0.0056 m		
ΔX:	0.0002 m		
ΔY:	-0.0004 m	USPA to Pseudo	968 m
ΔZ:	0.0014 m		
ΔX:	0.0014 m		
ΔY:	0.0040 m	USPB to Reilly	1,252 m
ΔZ:	-0.0033 m		
ΔX:	0.0009 m		
ΔY:	0.0015 m	Bromilow to Reilly	112 m
ΔZ:	-0.0010 m		

The **Σ** matrix is 21 x 21 and composed of seven 3 x 3 sub matrices – one from each baseline. Because we choose $\sigma_0^2 = 1.00$, the weight matrix is the inverse of the **Σ** matrix. The weight matrix is shown on the following page.

$$
\begin{bmatrix}
6.321E-06 & 1.546E-05 & -1.061E-05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.546E-05 & 4.740E-05 & -3.185E-05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.061E-05 & -3.185E-05 & 2.388E-05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.412E-06 & 1.285E-06 & -5.669E-07 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.285E-06 & 4.653E-06 & -1.658E-06 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -5.669E-07 & -1.658E-06 & 1.872E-06 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 9.505E-08 & 8.957E-08 & -5.022E-08 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 8.957E-08 & 3.729E-07 & -2.222E-07 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -5.022E-08 & -2.222E-07 & 3.364E-07 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.650E-07 & 9.024E-07 & -6.189E-07 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9.024E-07 & 2.796E-06 & -1.881E-06 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6.189E-07 & 1.881E-06 & 1.410E-06 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.763E-07 & 3.200E-07 & -2.009E-07 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.200E-07 & 6.870E-07 & -4.006E-07 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.009E-07 & -4.006E-07 & 4.662E-07 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.326E-07 & 1.312E-07 & -7.253E-08 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.317E-07 & 5.265E-07 & -3.021E-07 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7.253E-08 & -3.021E-07 & 5.007E-07 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.368E-07 & 3.937E-07 & -5.187E-07 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.937E-07 & 8.766E-07 & -8.978E-07 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.187E-07 & -8.978E-07 & 1.446E-06
\end{bmatrix}
$$

The weight matrix, **W**, is also 21 x 21 and composed of seven 3 x 3 sub matrices – one from each baseline. Note that with 0's on the remaining off-diagonal elements, each individual sub matrix in the weight matrix is the inverse of the corresponding submatrix in the **Σ** matrix.

It is commonly known that the statistics for the baseline vectors are often overstated by the various manufacturers. That really is of little consequence because the user can scale the weights at will by picking any desired value for σ_0^2.

794,441	-210,579	72,233	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-210,579	258,851	251,628	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
72,233	251,628	409,559	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	954,092	-234,682	81,047	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-234,682	371,707	258,103	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	81,047	258,103	787,146	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	13,607,521	-3,393,197	-209,741	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-3,393,197	5,267,810	2,973,094	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-209,741	2,973,094	4,905,460	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	13,732,790	-3,678,202	1,120,432	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-3,678,202	4,471,172	4,350,814	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1,120,432	4,350,814	7,004,665	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	7,917,229	-3,404,340	486,222	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	-3,404,340	4,381,420	2,298,352	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	486,222	2,298,352	4,329,973	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10,040,559	-2,565,406	-93,324	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2,565,406	3,560,587	1,776,790	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-93,324	1,776,790	3,055,967	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7,426,129	-1,670,110	1,626,106
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1,670,110	3,506,200	1,577,349
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1,626,106	1,577,349	2,253,380

ΔX: 0.0000 m ΔY: -0.0015 m ΔZ: 0.0027 m	Pseudo to Reilly	314 m
ΔX: -0.0011 m ΔY: -0.0020 m ΔZ: 0.0023 m	Bromilow to Pseudo	206 m

Using these residuals, the estimated (a posteriori) reference variance is:

$$(\text{sigma}_0 \text{ hat})^2 = \mathbf{v}^t \mathbf{W} \mathbf{v} \ / \ r = 115.2052 / 9 = 12.8006$$

Next, the standard deviation of each X/Y/Z coordinate is obtained from $(\text{sigma}_0 \text{ hat})^2 \mathbf{N}^{-1}$ where the variance of each parameter is computed from the diagonal element of $\mathbf{N}^{-1}$. This time, $\mathbf{N}^{-1}$ is fully populated and there are no 0's on the off-diagonals. Only the diagonal elements are used to compute the standard deviations of the parameters.

USPA	$\sigma_X = \sqrt{(12.8006 * 1.69E-07)} = 0.0015$ m $\sigma_Y = \sqrt{(12.8006 * 6.62E-07)} = 0.0029$ m $\sigma_Z = \sqrt{(12.8006 * 5.32E-07)} = 0.0026$ m
USPB	$\sigma_X = \sqrt{(12.8006 * 2.43E-07)} = 0.0018$ m $\sigma_Y = \sqrt{(12.8006 * 1.70E-06)} = 0.0047$ m $\sigma_Z = \sqrt{(12.8006 * 8.55E-07)} = 0.0033$ m
Pseudo	$\sigma_X = \sqrt{(12.8006 * 9.71E-08)} = 0.0011$ m $\sigma_Y = \sqrt{(12.8006 * 3.52E-07)} = 0.0021$ m $\sigma_Z = \sqrt{(12.8006 * 3.27E-07)} = 0.0020$ m
Bromilow	$\sigma_X = \sqrt{(12.8006 * 1.62E-07)} = 0.0014$ m $\sigma_Y = \sqrt{(12.8006 * 4.34E-07)} = 0.0024$ m $\sigma_Z = \sqrt{(12.8006 * 3.65E-07)} = 0.0022$ m

As before, the X/Y/Z position of each new point is used to compute the latitude and longitude of the point. Then, those geodetic positions are used to rotate the X/Y/Z reference frame standard deviations to the local reference frame. However, this time the entire covariance matrix of the computed position is used instead of just the standard deviations derived from the diagonal elements.

The covariance matrix of the computed position is also used as the basis for computing local and network accuracies of the inverse distance and direction between points and is described in a subsequent section. Both the $\mathbf{N}^{-1}$ matrix and the covariance matrix of the computed positions for points USPA, USPB, Pseudo, and Bromilow [$(\text{sigma}_0 \text{ hat})^2 * \mathbf{N}^{-1}$] are shown on the following page.

N^{-1} as printed out from the least squares adjustment program is:

1.688E-7	1.834E-7	-1.169E-7	3.401E-8	8.217E-8	-5.782E-8	8.910E-8	1.004E-7	-6.974E-8	4.530E-8	5.233E-8	-2.315E-8
1.834E-7	6.620E-7	-3.920E-7	8.806E-8	2.747E-7	-1.918E-7	1.010E-7	3.312E-7	-2.012E-7	5.318E-8	1.559E-7	-7.339E-8
-1.169E-7	-3.920E-7	5.321E-7	-8.636E-8	-2.657E-7	2.100E-7	-6.898E-8	-2.006E-7	2.815E-7	-1.119E-8	-5.992E-8	7.011E-8
3.401E-8	8.806E-8	-8.636E-8	2.433E-7	5.443E-7	-3.649E-7	1.897E-8	4.519E-8	-4.598E-8	7.051E-9	1.778E-8	-1.296E-8
8.217E-8	2.747E-7	-2.657E-7	5.443E-7	1.705E-6	-1.109E-6	4.674E-8	1.392E-7	-1.395E-7	1.595E-8	5.384E-8	-3.967E-8
-5.782E-8	-1.918E-7	2.100E-7	-3.649E-7	-1.109E-6	8.549E-7	-3.326E-8	-9.757E-8	1.107E-7	-9.419E-9	-3.494E-8	2.987E-8
8.910E-8	1.010E-7	-6.898E-8	1.897E-8	4.674E-8	-3.326E-8	9.706E-8	1.049E-7	-7.197E-8	4.955E-8	5.538E-8	-2.402E-8
1.004E-7	3.312E-7	-2.006E-7	4.519E-8	1.392E-7	-9.757E-8	1.049E-7	3.521E-7	-2.168E-7	5.477E-8	1.647E-7	-7.849E-8
-6.974E-8	-2.012E-7	2.815E-7	-4.598E-8	-1.395E-7	1.107E-7	-7.197E-8	-2.168E-7	3.267E-7	-7.207E-9	-5.946E-8	7.943E-8
4.530E-8	5.318E-8	-1.119E-8	7.051E-9	1.595E-8	-9.419E-9	4.955E-8	5.477E-8	-7.207E-9	1.626E-7	1.849E-7	-1.381E-7
5.233E-8	1.559E-7	-5.992E-8	1.778E-8	5.384E-8	-3.494E-8	5.538E-8	1.647E-7	-5.946E-8	1.849E-7	4.339E-7	-2.820E-7
-2.315E-8	-7.339E-8	7.011E-8	-1.296E-8	-3.967E-8	2.987E-8	-2.402E-8	-7.849E-8	7.943E-8	-1.381E-7	-2.820E-7	3.647E-7
	USPA			**USPB**			**Pseudo**			**Bromilow**	

The covariance matrix of the computed points (USPA, USPB, Pseudo, and Bromilow) is (sigma$_0$ hat)^2 * **N**$^{-1}$ and equals:

2.161E-6	2.347E-6	-1.496E-6	4.354E-7	1.052E-6	-7.402E-7	1.141E-6	1.285E-6	-8.927E-7	5.799E-7	6.698E-7	-2.964E-7
2.347E-6	8.474E-6	-5.017E-6	1.127E-6	3.517E-6	-2.456E-6	1.293E-6	4.240E-6	-2.576E-6	6.807E-7	1.995E-6	-9.394E-7
-1.496E-6	-5.017E-6	6.812E-6	-1.105E-6	-3.401E-6	2.689E-6	-8.829E-7	-2.568E-6	3.603E-6	-1.432E-7	-7.670E-7	8.975E-7
4.354E-7	1.127E-6	-1.105E-6	3.114E-6	6.968E-6	-4.671E-6	2.428E-7	5.785E-7	-5.886E-7	9.026E-8	2.276E-7	-1.659E-7
1.052E-6	3.517E-6	-3.401E-6	6.968E-6	2.182E-5	-1.420E-5	5.983E-7	1.782E-6	-1.786E-6	2.042E-7	6.892E-7	-5.078E-7
-7.402E-7	-2.456E-6	2.689E-6	-4.671E-6	-1.420E-5	1.094E-5	-4.258E-7	-1.249E-6	1.416E-6	-1.206E-7	-4.472E-7	3.824E-7
1.141E-6	1.293E-6	-8.829E-7	2.428E-7	5.983E-7	-4.258E-7	1.242E-6	1.343E-6	-9.212E-7	6.343E-7	7.089E-7	-3.075E-7
1.285E-6	4.240E-6	-2.568E-6	5.785E-7	1.782E-6	-1.249E-6	1.343E-6	4.506E-6	-2.775E-6	7.011E-7	2.109E-6	-1.005E-6
-8.927E-7	-2.576E-6	3.603E-6	-5.886E-7	-1.786E-6	1.416E-6	-9.212E-7	-2.775E-6	4.182E-6	-9.225E-8	-7.611E-7	1.017E-6
5.799E-7	6.807E-7	-1.432E-7	9.026E-8	2.042E-7	-1.206E-7	6.343E-7	7.011E-7	-9.225E-8	2.081E-6	2.367E-6	-1.768E-6
6.698E-7	1.995E-6	-7.670E-7	2.276E-7	6.892E-7	-4.472E-7	7.089E-7	2.109E-6	-7.611E-7	2.367E-6	5.554E-6	-3.609E-6
-2.964E-7	-9.394E-7	8.975E-7	-1.659E-7	-5.078E-7	3.824E-7	-3.075E-7	-1.005E-6	1.017E-6	-1.768E-6	-3.609E-6	4.668E-6
	USPA			**USPB**			**Pseudo**			**Bromilow**	

	Geocentric & ECEF sigma		Geodetic & local sigma	

Station USPA:

X =	-1,555,678.579 m	+/- 0.0015 m	φ = 32º 16' 23."00019 N	+/- 0.0027 m	(N)
Y =	-5,169,961.396 m	+/- 0.0029 m	λ = 106º 44' 48."90817 W	+/- 0.0017 m	(E)
Z =	3,386,700.089 m	+/- 0.0026 m	h = 1,178.015 m	+/- 0.0028 m	(U)

Station USPB:

X =	-1,555,663.613 m	+/- 0.0018 m	φ = 32º 16' 22."36244 N	+/- 0.0037 m	(N)
Y =	-5,169,976.761 m	+/- 0.0047 m	λ = 106º 44' 48."19151 W	+/- 0.0022 m	(E)
Z =	3,386,683.419 m	+/- 0.0033 m	h = 1,177.908 m	+/- 0.0042 m	(U)

Station Pseudo:

X =	-1,556,206.615 m	+/- 0.0011 m	φ = 32º 16' 45."74650 N	+/- 0.0020 m	(N)
Y =	-5,169,400.740 m	+/- 0.0021 m	λ = 106º 45' 14."39975 W	+/- 0.0012 m	(E)
Z =	3,387,285.987 m	+/- 0.0020 m	h = 1,165.641 m	+/- 0.0020 m	(U)

Station Bromilow:

X =	-1,556,209.750 m	+/- 0.0014 m	φ = 32º 16' 52."33407 N	+/- 0.0022 m	(N)
Y =	-5,169,286.496 m	+/- 0.0024 m	λ = 106º 45' 15."77273 W	+/- 0.0015 m	(E)
Z =	3,387,457.512 m	+/- 0.0022 m	h = 1,165.523 m	+/- 0.0023 m	(U)

Note that the adjusted X/Y/Z coordinate values changed slightly but that the standard deviations, both in the X/Y/Z reference frame and the local components, are larger – not smaller as might have been expected. In order to understand that counter-intuitive result, we need to look at the difference between network accuracy and local accuracy.

Network Accuracy and Local Accuracy

Datum accuracy, network accuracy, and local accuracy are defined mathematically in the article, "Spatial Data Accuracy as Defined by the GSDM" Journal of Surveying and Land Information Systems, Vol. 59, No. 1, March, 1999, pp 26-30. Datum accuracy is a statement of how well the position of a single point is known with respect to the published datum. Network accuracy can be intuitively understood to be a statement of accuracy between points based upon how well the positions are known with respect to the control held by the user. It is presumed the points are independent – that is, there is no correlation of one with respect to the other as might be determined by a direct tie between them. Alternatively, local accuracy can be understood to be a statement of accuracy between points based upon a direct measurement between the points. The following paragraphs describe the results of computing both network accuracy and network accuracy from point USPA to point Pseudo. An Excel spreadsheet (the file is called "3-D inverse with statistics.xls") was used to generate the answers and can be obtained gratis from the author at globalcogo@zianet.com.

When using the Excel spreadsheet, the user keys information into the spreadsheet and answers appear instantaneously. Input includes the names of the two stations, the geocentric X/Y/Z coordinates of the two points, and the standard deviation (really covariance) information. When computing the inverse, the direction and distance will remain the same but the standard deviations will be different depending upon the covariance information input by the user. Choices for entering covariance information are:

1. All standard deviations are entered as zeros. That means there is no standard deviation available and the X/Y/Z coordinate data are used as being "fixed." The spreadsheet will still compute the local tangent plane direction and distance between points, but there will be no standard deviations associated with the inverse direction and distance.

2. The user can enter the standard deviations of the geocentric X/Y/Z coordinates as variances (standard deviations squared). These covariance data are entered on the diagaonal of the geocentric covariance matrix for each point. The spreadsheet computes the local reference frame covariance matrix (showing the local component e/n/u standard deviations of each point), the inverse direction and distance standpoint to forepoint, and the standard deviation of the direction and the distance. Local and network accuracy will be identical because no correlation data were entered.

3. The user can enter the full covariance matrix for each point. This is the "best" inverse one can get without also providing correlation information. This answer is "network" accuracy and presumes the coordinates of the two points are statistically independent of one another. Local accuracy will compute as being identical to network accuracy.

4. Or, the user may enter the full covariance matrix at each point as well as the correlation matrices between points. The correlation of the Forepoint with respect to the Standpoint is the transpose of the correlation of the Standpoint with respect to the Forepoint. It is redundant, but both correlation matrices need to be entered (the asute Excel user will quickly rekey the appropriate cells so that correlation data needs to be entered only once).

Values in the comparison below were computed using the file 3-D inverse with statistics.xls.

	Inverse - USPA to Pseudo				Network Accuracy	Local Accuracy
1.	No standard deviations	Distance =	967.615 m	+/-	0.0000 m	0.0000 m
		Direction =	316° 24' 28."2	+/-	0.00 sec.	0.00 sec.
2.	Standard deviations of	Distance =	967.615 m	+/-	0.0031 m	0.0031 m
	X/Y/Z values only	Direction =	316° 24' 28."2	+/-	0.53 sec	0.53 sec.
3.	Full covariance matrix	Distance =	967.615 m	+/-	0.0018 m	0.0018 m
	of each X/Y/Z point	Direction =	316° 24' 28."2	+/-	0.40 sec.	0.40 sec.
4.	Full covariance matrix	Distance =	967.615 m	+/-	0.0018 m	0.0011 m
	and correlation submatrix	Direction =	316° 24' 28."2	+/-	0.40 sec.	0.24 sec.