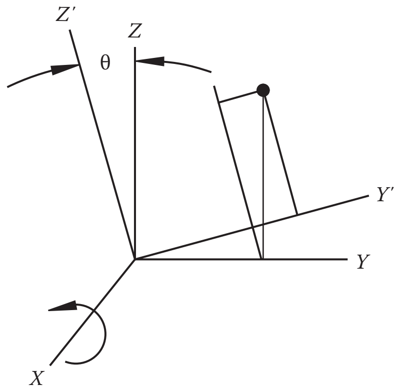
Appendix A: Rotation Matrix Derivation

A rotation matrix is a collection of equations expressed in matrix form and used to change the perspective associated with spatial data. In the case of the GSDM, the perspective changes from looking at points in the geocentric ECEF reference frame to viewing the same data from the perspective of one occupying a given point (called the point of beginning—P.O.B.) and viewing all other points as if standing at the selected origin.

One way of applying the rotation matrix is to recompute the coordinates of each point with respect to the original origin. Another approach is to apply the rotation matrix to the vector from the P.O.B. to any other point in the database. The GSDM uses the second approach. That means the geocentric coordinate differences are found first. Then the rotation matrix is applied to the ECEF vector components and the result is local "flat-Earth" components from the P.O.B. to the selected point. Although it could be argued that the requirement to compute values of the rotation matrix for each P.O.B. is a disadvantage, several advantages are as follows:

1. The underlying primary data (ECEF coordinates) are not modified and remain available for immediate recomputation—such as selecting a different P.O.B.
2. After using the rotation matrix, the local components of any vector appear to the user as rectangular flat-Earth plane surveying components. These components can be used in a variety of operations as selected by the user.
3. The process is reversible in that local perspective components can also be rotated into the ECEF perspective, making such differences compatible with *X*/*Y*/*Z* points already stored in the database. The *X*/*Y*/*Z* coordinates of new points are established using simple addition and subtraction operations.
4. If the spatial data user needs more traditional values for a point, such as latitude/longitude or state plane coordinates, those values are also immediately available by rigorous computation (though not involving a rotation matrix) from the geocentric ECEF values for the specified point as stored in the database.

The convention used is that a positive rotation is counterclockwise as viewed looking at the origin from the positive direction along the axis being rotated (Leick 2004). This rule needs to be applied once for each of the three possible axes. This process yields three separate matrices—one for each rotation. This part is generic without regard to being attached to the Earth and each rotation is illustrated in a separate



$X' = X$

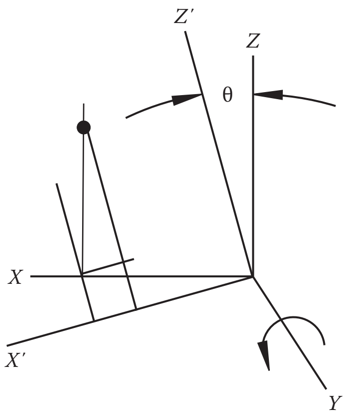
$Y' = Y\cos\theta + Z\sin\theta$

$Z' = -Y\sin\theta + Z\cos\theta$

In matrix form,

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \text{(A.1)}$$

FIGURE A.1 R_1 Rotation about the *X*-Axis



$X' = X\cos\theta - Z\sin\theta$

$Y' = Y$

$Z' = X\sin\theta + Z\cos\theta$

(A.1)

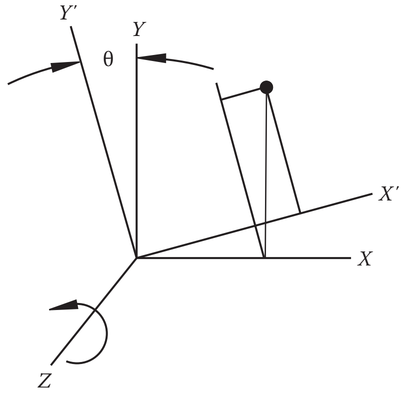
In matrix form,

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \text{(A.2)}$$

FIGURE A.2 R_2 Rotation about the *Y*-Axis

diagram. Figure A.1 shows a positive rotation (R_1) about the *X*-axis. Figure A.2 shows a positive rotation (R_2) about the *Y*-axis. And, Figure A.3 shows a positive rotation (R_3) about the *Z*-axis.

With each rotation quantified, the process is applied to vectors (coordinate differences) attached to the Earth. Starting with the *X*/*Y*/*Z* axes of the ECEF reference frame, the first rotation is a positive rotation about the *Z*-axis (R_3) to bring the *Y*-axis into the vertical plane of the local meridian. The angular amount is determined by the longitude of the selected P.O.B. and is computed as east longitude +90°. A full circle can be subtracted if the sum goes over 360°. The second rotation is also a positive rotation about *X*-axis (R_1) to bring the *Y*-axis into the local geodetic horizon. The two rotations describe movement of the original *Y*-axis in two rotations—first is a positive rotation about the *Z*-axis, then a second positive rotation about the *X*-axis. Other rotation sequences could be used to obtain the same result. The original $\Delta X/\Delta Y/\Delta Z$



$X' = X\cos\theta + Y\sin\theta$

$Y' = -X\sin\theta + Y\cos\theta$

$Z' = Z$

In matrix form,

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \text{(A.3)}$$

FIGURE A.3 R_3 Rotation about the Z-Axis

vector is right handed and the rotated differences are also right handed if used as $\Delta e/\Delta n/\Delta u$.

Multiplication of the two matrices is shown below. The rules of matrix multiplication require the sequence to be as shown in Equation A.4. In Equation A.5, the actual latitude and longitude of the selected P.O.B. are used and Equation A.6 is a simplification based upon making substitutions for trigonometric identities. Finally, Equation A.7 is the form of the rotation matrix given in Equation 1.21.

It is stated without proof here that Equation 1.22 uses the transpose of the rotation matrix to convert the local geodetic horizon perspective to the ECEF perspective. Being able to use the transpose for the reverse computation is a consequence of the two right-handed systems, both being orthogonal—see Chapter 3 (Vanicek and Krakiwsky 1986).

Now, using the matrices in Figures A.1 and A.3 above, the process is attached to the Earth in the following matrix statement:

$$\begin{bmatrix} \Delta e \\ \Delta n \\ \Delta u \end{bmatrix} = R_1\left(90^\circ - \phi\right) R_3\left(\lambda + 90^\circ\right) \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \quad \text{(A.4)}$$

$$\begin{bmatrix} \Delta e \\ \Delta n \\ \Delta u \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\left(90^\circ-\phi\right) & \sin\left(90^\circ-\phi\right) \\ 0 & -\sin\left(90^\circ-\phi\right) & \cos\left(90^\circ-\phi\right) \end{bmatrix} \begin{bmatrix} \cos\left(\lambda+90^\circ\right) & \sin\left(\lambda+90^\circ\right) & 0 \\ -\sin\left(\lambda+90^\circ\right) & \cos\left(\lambda+90^\circ\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \quad \text{(A.5)}$$

$$\begin{bmatrix} \Delta e \\ \Delta n \\ \Delta u \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\phi & \cos\phi \\ 0 & -\cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\cos\lambda & -\sin\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \quad \text{(A.6)}$$

$$\begin{bmatrix} \Delta e \\ \Delta n \\ \Delta u \end{bmatrix} = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\phi & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \tag{A.7}$$

REFERENCES

Leick, A. 2004. *GPS Satellite Surveying*, 3rd ed. New York: John Wiley & Sons.
Vanicek, P. and Krakiwsky, E. 1986. *Geodesy: The Concepts*. New York: North Holland.